Model theory for metric structures

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Resumen

La lógica de primer orden continua es una generalización de la clásica donde el conjunto de valores de verdad $\{0, 1\}$ se substituye por un continuo. Una *estructura métrica* es una estructura de varias clases donde cada clase es un espacio métrico completo de diámetro acotado, junto con: elementos distinguidos (pertenecientes a los diferentes espacios métricos) y funciones uniformemente continuas, o bien entre las clases o de las clases a un intervalo acotado de $\mathbb{R}$.

El trabajo consite en un desarrollo de los principales conceptos y resultados de la teoría de modelos para estructuras métricas. Se basa en los artículos *Model theory for metric structures* de I. Ben Yaacov, et al. y *Algebraic closure in continuous logic* de C. W. Henson, y H. Tellez, Desarrollamos la construcción del ultraproducto de estructuras métricas y probamos el teorema de Łoś's y el teorema de compacidad. Probamos el teorema de Lowenheim-Skolem y la existencia de estructuras suficientemente saturadas y fuertemente homogéneas. Demostramos que el espacio de tipos es un espacio topológico metrizable. Consideramos diferentes conceptos de definabilidad y algebraicidad de los que damos varias cracterizaciones y estudiamos como se comportan en extensiones y subestructuras elementales. Finalmente consideramos la teoría de los espacios de Hilbert en este contexto, en particular los de dimensión infinita, caracterizamos la clausura definible de un conjunto usando la clausura respecto a la norma de las combinaciones lineales de elementos del conjunto y probamos que el tipo de una tupla sobre un conjunto A está definido por su proyección sobre el subespacio generado por A y el producto escalar de las coordenadas, relacionamos la métrica del espacio de tipos con la métrica del espacio de Hilbert.

Abstract

Continuous first order logic is a generalization of classical first order logic where a continuum is allowed as truth value set. A *metric structure* is a many-sorted structure where each sort is a complete metric space of bounded diameter, together with distinguished elements (belonging to the distinct sorts) and uniformly continuous functions, either between sorts or from the sorts into bounded closed intervals of $\mathbb{R}$.

We develop the main concepts of model theory for metric structures. This memoir is based on *Model theory for metric structures* by I. Ben Yaacov, et al. and *Algebraic closure in continuous logic* by C. W. Henson, and H. Tellez, We develop the ultraproduct of metric structures and prove Łoś's theorem and compactness theorem. We prove Löwenheim-Skolem theorem and the existence of sufficiently saturated and strongly homogeneous structures. We prove that the type space is a metrizable topological space. We introduce several concepts of definability and algebraicity, we prove some characterization results and study their behaviour in elementary extensions and substructures. Finally, we consider the theory of infinite Hilbert spaces in this context, we characterize the definable closure of a set using the norm closure and linear span of the set and prove that the type of an tuple over a set A is defined by its projection over the subspace generated by A and the inner product of the coordinates. We show the relation between the metric on the space of types and the metric of the Hilbert space.

Contents

Introduction

Spanish

En la lógica clásica de primer orden, a cada enunciado se le asigna un valor de verdad 0 o 1. La lógica continua de primer orden es una generalización donde el conjunto de valores $\{0, 1\}$ se substituye por un conjunto más complejo, en este trabajo va a ser el intervalo real [0,1]. Una *estructura métrica* es una estructura de varias clases donde cada una de las clases es un espacio métrico completo de diámetro acotado. Las estructuras métricas pueden también tener elementos distinguidos pertenecientes a los diferentes espacios métricos (constantes) y funciones uniformemente continuas o bien entre las clases (funciones) o de las clases a un intervalo acotado de $\mathbb{R}$ (predicados). Por simplicidad, es útil asumir que todos los intervalos acotados son el intervalo $[0, 1]$. Algunos ejemplos de estructuras métricas son los espacios métricos, los retículos de Banach, las C^*-álgebras, los espacios de Hilbert y las estructuras de la teoría de modelos clásica. En la lógica continua, la métrica d (de cada clase) ocupa el lugar del símbolo $=$ en la lógica clásica. El requisito de continuidad uniforme sobre las funciones y predicados es esencial para poder desarrollar una buena teoría. En la lógica continua, una conectiva n-aria es cualquier función continua de $[0, 1]^n$ en $[0, 1]$. Sin embargo, esta definición de conectiva puede extenderse a funciones continuas de $[0, 1]^{\mathbb{N}}$ en $[0, 1]$ (véase proposición 2.1.2). Puesto que el conjunto de valores de verdad que consideramos está linealmente ordenado, es natural que dos cuantificadores importantes sean sup e inf. La mayoría de los resultados de la teoría de modelos clásicas tienen un análogo en la teoría de modelos para estructuras métricas, de hecho, varios de los resultados de la teoría de modelos clásica se pueden obtener restringiendo los resultados para estructuras métricas al caso donde la métrica d es la métrica discreta.

La teoría de modelos para espacios métricos aparece por primera vez en 1966 en el libro *Continuous Model Theory* de C. C. Chang y H. J. Keisler [8], los autores permiten cualquier conjunto Hausdorff compacto como conjunto de valores de verdad. El desarrollo de esta teoría fue retomado más tarde por C. W. Henson [12], [13] basado los trabajos de J. L. Krivine [17], [18] y Stern [21], más tarde por J. Iovino [16] y más recientemente por I. Ben Yaacov [3], A. Usvyatsov [22], M. Lupini [11] y otros autores [7]. Actualmente, la teoría de modelos para espacios métricos es un área en auge y con perspectivas de futuro, existen una gran cantidad de publicaciones recientes con importantes resultados de diferentes grupos de investigación (véase por ejemplo [11], [7], [4], [6] y [23]).

Una motivación adicional para estudiar la teoría de modelos para estructuras métricas son sus aplicaciones en análisis, análisis funcional [14] y geometría [20]. Estas aplicaciones suelen estar relacionadas con ultraproductos de estructuras métricas, aunque otras lógicas han sido también desarrolladas para estudiar estas aplicaciones [1].

Como aplicación de la teoría de modelos para espacios métricos cabe mencionar el la demostración de Ben Yaacov en [2] de que el grupo de las isometrías lineales del espacio de Gurarij es un grupo polaco universal, donde el espacio de Gurarij es el único espacio de Banach separable, universal y aproximadamente homogéneo.

Este trabajo está basado principalmente en los artículos *Model theory for metric structures* [5] y *Algebraic closure in continuous logic* [15]. Puesto que ambos son artículos extensos donde se desarrolla la teoría desde el principio y con mucho detalle, varias demostraciones se han extraído literalmente para hacer este trabajo autocontenido en la medida de lo posible. Las demostraciones más avanzadas y con menos detalles se han ampliado y completado. La memoria se divide en tres capítulos. En el primer capítulo introducimos los conceptos básicos de la teoría de modelos para estructuras métricas, entre estos conceptos se encuentra el de estructura métrica, preestructura métrica e inmersión. Demostramos algunos resultados preliminares como el análogo al test de Tarski-Vaught (proposición 1.1.5) . También tratamos un problema relacionado con la cardinalidad del conjunto de fórmulas que surje al permitir cualquier función continua de $[0,1]$ en $[0,1]$ como conectiva. Presentamos la construcción del ultraproducto de estructuras métricas y demostramos los análogos a resultados tales como el teorema fundamental de los ultraproductos (teorema 1.2.7) y el teorema de compacidad (teorema 1.2.11). Presentamos también una caracterización de cuando una clase de estructuras métricas es axiomatizable. Demostramos el teorema de Löwenheim-Skolem (proposición 1.2.17) además de la existencia de estructuras suficientemente saturadas y fuertemente homogéneas (teorema 1.2.23). Finalizamos el primer capítulo con la construcción del espacio de tipos, probamos que en el contexto de la teoría de modelos para estructuras métricas es un espacio topológico metrizable (teorema 1.3.7). En el segundo capítulo introducimos los conceptos de predicado definible, conjunto definible y función definible, estudiamos como se comportan estos conceptos con respecto a extensiones y subestructuras elementales, demostramos que podemos axiomatizar los predicados con forma $dist(x, D)$, donde D es un conjunto cerrado (teorema 2.1.8) y demostramos varias caracterizaciones de definibilidad, por ejemplo, en una estructura suficientemente saturada una función es definible si y solo si su grafo es un conjunto tipo-definible (proposición 2.1.14). Concluimos el segundo capítulo con las definiciones de clausura algebraica y clausura definible, estudiamos su comportamiento en extensiones y subestructuras elementales además de probar varios resultados de caracterización de elementos algebraicos y definibles, por ejemplo, un elemento de una estructura métrica es definible si y solo si en cualquier extensión elemental de esta estructura no existen otros elementos que realicen el mismo tipo (proposición 2.2.6). En el tercer capítulo, usamos los conceptos y resultados desarrollados a lo largo del trabajo para estudiar la teoría de los espacios de Hilbert infinito-dimensionales IHS. Demostramos que la clausura definible de un conjunto es la clausura con respecto de la norma de las combinaciones lineales de elementos del conjunto (proposición 3.0.2). Probamos que el tipo de una tupla sobre un conjunto A está definido por la proyección de las coordenadas sobre el subespacio generado por A y el producto de las coordenadas dos a dos (lema 3.0.1). Terminamos presentando algunos resultados adicionales sobre la teoría IHS.

English

In classical first order logic, a truth value of 0 or 1 is assigned to each sentence. Continuous first order logic is a generalization where more complex sets are allowed as truth value sets, in this memoir the set of truth values will be the real interval $[0, 1]$. A metric structure is a many-sorted structure where each of the sorts is a complete metric space of bounded diameter. Metric structures can also have distinguished elements belonging to the distinct sorts (constants) and uniformly continuous functions between sorts (functions) and from the sorts into bounded closed intervals of $\mathbb{R}$ (predicates), for convenience, it is useful to assume that all intervals are $[0, 1]$. Some examples of metric structures are metric spaces, Banach lattices, C^*-algebras, Hilbert spaces and structures in the sense of classical model theory. In continuous first order logic, the metric d (of each sort) plays the role of the symbol $=$ in the classical case. The uniform continuity of the functions and predicates is essential to develop a successful theory. In continuous first order logic, the n-ary connectives are continuous functions from $[0, 1]^n$ into $[0, 1]$. However, one could broaden the definition of connective to allow continuous functions from $[0, 1]^{\mathbb{N}}$ into $[0, 1]$ (see proposition 2.1.2). As our set of truth values is linearly ordered, is natural that we have two special quantifiers, sup and inf. Most of the results of classical model theory have an analogous counterpart in model theory for metric structures, furthermore, sometimes the results in the metric structures setting imply the classical results when we consider the metric to be discrete metric.

Model theory for metric structures was first introduced in 1966 in the book *Continuous Model Theory* by C. C. Chang y H. J. Keisler [8], the authors allowed any compact Hausdorff space as a set of truth values. The development of the theory was retaken by C. W. Henson [12], [13] based on the publications of J. L. Krivine [17], [18] and Stern [21], later by J. Iovino [16] y and more recently by I. Ben Yaacov [3], A. Usvyatsov [22], M. Lupini [11] and other authors [7]. Nowadays, model theory for metric structures if a flourishing topic with good prospects for the future. There exists a large number of recent publications with significant results from different researching groups (see [11], [7], [4], [6] and [23] for example).

Other motivations to study model theory for metric structures are its connection to applications in analysis, functional analysis and geometry [20]. These applications are usually based in the ultraproduct construction [14]. Other logics have also been used to study these applications [1].

As an explicit application of model theory for metric structures, we mention the proof of Ben Yaacov in [2] of the linear isometry group of the Gurarij space being an universal Polish group, where the Gurarij space is the unique, separable, universal, approximately homogeneous Banach space.

This memoir is mainly based in the papers *Model theory for metric structures* [5] and *Algebraic closure in continuous logic* [15]. Since both of them are papers where the theory is developed from the beginning and in great detail, several proofs have been literally drawn to make this memoir as self-contained as possible. Those proofs that were more advanced and those that were less detailed have been expanded and more details have been added. The memoir is divided in three chapters. In the first chapter, we introduce the basics concepts of model theory for metric structures as structures, prestructures and embeddings, we also prove some preliminary results as the analogous of the Tarski-Vaught test (proposition 1.1.5). We discuss the cardinality problem of the set of formulas that arise when we allow all continuous functions on $[0, 1]$ to be connectives. We show the construction of the ultraproduct of metric structures and prove results as the analogous of the fundamental theorem of ultraprodutcs (Łoś's Theorem 1.2.7), the compactness theorem (theorem 1.2.11), and a characterization of the axiomatizability of a class of metric structures. We prove Löwenheim-Skolem theorem (proposition 1.2.17) and the existence of sufficiently saturated and strongly homogeneous models (theorem 1.2.23). The first chapter ends with the construction of the space of types, we show that in the metric structures setting, the space of types is a metrizable topological space (theorem 1.3.7). In the second chapter we introduce the concepts of definable predicates, definable sets and definable functions. We study their behaviour with respect to elementary extensions and substructures, we prove that predicates of the form $dist(x, D)$, where D is a closed set, are axiomatizable (theorem 2.1.8) and we prove several characterizations of these objects, for example, in sufficiently saturated structures a function is definable if and only if its graph is a type-definable set (proposition 2.1.14). The second chapter ends with the definable and algebraic closures, we study their behaviour in elementary extensions and substructures. We also prove some results about characterizations of definable and algebraic elements, for example, an element of a metric structure is definable if and only if is the only realization of its type in any elementary extension o f the structure (proposition 2.2.6). In the last chapter, we apply the results and concepts developed previously to the theory of infinite dimensional Hilbert spaces IHS. We characterize the definable closure of a set using the norm closure and linear span of the set (proposition 3.0.2). We prove a results that shows that the type of an tuple over a set A is defined by its projection over the subspace generated by A and the inner product of the coordinates (lemma 3.0.1). The memoir ends introducing further results about the theory IHS.

CHAPTER 1
Metric Structures

We begin by giving the basic definitions needed to develop the model theory of metric structures. As one can see as one reads, in general, these definitions are very similar to the classical ones. We will try to emphasize the differences between these two theories.

1.1. Basics

Let (M, d) be a complete, bounded metric space. A *metric structure* $\mathcal{M}$ based on (M, d) is a tuple

$$\mathcal{M} = (M, R_i, F_j, a_k : i \in I, j \in J, k \in K).$$

Where each R_i is an uniformly continuous function from M^n for some $n \geqslant 1$ into some bounded interval in $\mathbb{R}$, a *predicate*. Each F_j is an uniformly continuous function from M^n for some $n \geqslant 1$ into M, we call it *function* or *operation*. And each a_k is a distinguished element of M, a *constant*. Sometimes, d will be treated as a binary predicate, and expressions like $x = y$ will be used instead of $d(x, y) = 0$. If all the index sets are empty, $\mathcal{M}$ is just a bounded, complete metric space.

To motivate this work, we give some examples of metric structures that could be studied with the machinery we are going to develop:

A bounded, complete metric space.

The unit ball of a Banach space X over $\mathbb{C}$ or $\mathbb{R}$, where the norm is included as a predicate, the element 0 as a constant and the functions are $f_{\alpha,\beta}(x, y) = \alpha x + \beta y$ for α and β scalars satisfying $|\alpha| + |\beta| \leqslant 1$. To be more specific one can think of some $L^p(0, 1)$ with $p \in [1, \infty]$.

Remark 1.1.1. *We can look at structures of classical first order model theory as structures on this new logic. To do so, we endorse the universe A of an structure $\mathfrak{A}$ with the discrete metric. Functions of the structure are obviously uniformly continuous with respect to the discrete metric. Constant are also carried without changes. To bring relations to this new interpretation, we consider the set R of all elements that satisfy the relation $\mathcal{R}$ and we introduce the indicator function of the set R.*

As in classical first order logic, we will need the notion of language, or equivalently, the notion of *signature* of a metric structure. To each metric structure $\mathcal{M}$ we associate a *signature* L in a very similar way as one does in the classical theory. To each predicate R of $\mathcal{M}$ we associate a *predicate symbol* P and its arity; we denote R by $P^{\mathcal{M}}$. To each function F of $\mathcal{M}$ we associate a *function symbol* f and its arity. Finally, to each constant a of $\mathcal{M}$ we associate a *constant symbol* c; we denote a by $c^{\mathcal{M}}$. But for metric structures a signature must specify more: for each predicate symbol P, it must provide a closed bounded interval $I_P \subset \mathbb{R}$ where $P^{\mathcal{M}}$ takes its values, and a modulus of uniform continuity Δ_P for $P^{\mathcal{M}}$. For each function symbol f, L must provide a modulus of uniform continuity Δ_f for $f^{\mathcal{M}}$. Finally, L must provide a non negative real number D_L which is a bound on the complete metric space (M, d) on which $\mathcal{M}$ is based.

Thus, as in classical first order model theory, $\mathcal{M}$ is an L-*structure* if L correspond to the signature of $\mathcal{M}$.

For simplicity and without loss of generality, we will usually assume that our signatures L satisfy $D_L = 1$ and $I_P = [0, 1]$ for every predicate symbol P in L.

We introduce the definition of embedding of metric structures, the difference with the classical case is that we ask the embedding to be a metric space isometry, this is a natural requirement as one can see below.

Let L be a signature for metric structures and suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures. An *embedding* from $\mathcal{M}$ into $\mathcal{N}$ is a metric space isometry

$$T : (M, d^{\mathcal{M}}) \to (N, d^{\mathcal{N}})$$

that commutes with the interpretations of the function and predicate symbols of L as in the classical case We use the same notation and definition for the concepts of *isomorphism*, *automorphism* and *substructure* as we did in classical first order model theory.

We skip most of the construction of the syntactical part of the theory because it is the standard construction but we note the main differences with the classical one. Continuous functions $u : [0, 1]^n \to [0, 1]$ of finitely many variables $n \geqslant 1$ are the connectives and the symbols *sup* and *inf* are the quantifiers in this logic. Letting all continuous functions $u : [0, 1]^n \to [0, 1]$ to be connectives, could make the cardinality of the set of L-formulas too big. We treat this problem later in this section. Terms are constructed inductively, exactly as in classical first order-logic with (individual) variables and constants as terms of lowest complexity. However, formulas are a bit different. Fix a signature for metric structures, L, *atomic formulas* are formal expressions of the form $P(t_1, \ldots, t_n)$ or $d(t_1, t_2)$, where all $t_1, \ldots, t_n$ are L-terms and P is any predicate in L. The class of L-formulas, that we denote by $Form(L)$, is the smallest class that contains atomic formulas and is closed under the following rules.

1. If $u : [0, 1]^n \to [0, 1]$ is a connective and $\varphi_1, \ldots, \varphi_n$ are L-formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is an L-formula.

2. If φ is an L-formula and x is a variable, $\inf_x \varphi$ and $\sup_x \varphi$ are L-formulas.

Many other syntactic notions can be carried over word for word into this setting.

As in classical first order logic, the interpretation of an L-term t in $\mathcal{M}$, is a function $t^{\mathcal{M}} : M^n \to M$. However, the value of a $L(M)$-sentence σ is a real number in the interval $[0,1]$ and it is denoted $\sigma^{\mathcal{M}}$, which we are going to define, as usual, via the the value of formulas, which in turn we define by induction its complexity, as follows:

1. $(d(t_1,t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for any t_1, t_2;

2. for any n-ary predicate symbol P of L and $t_1,\ldots,t_n$,
$$(P(t_1\ldots,t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}},\ldots,t_n^{\mathcal{M}});$$

3. for any continuous function $u : [0,1]^n \to [0,1]$ and any $L(M)$-sentences $\sigma_1,\ldots,\sigma_n$,
$$(u(\sigma_1,\ldots,\sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}},\ldots,\sigma_n^{\mathcal{M}});$$

4. for any $L(M)$-formula $\varphi(x)$,
$$(\sup_x \varphi(x))^{\mathcal{M}}$$
is the supremum in $[0,1]$ of the set $\{\varphi(a)^{\mathcal{M}} : a \in M\}$;

5. for any $L(M)$-formula $\varphi(x)$,
$$(\inf_x \varphi(x))^{\mathcal{M}}$$
is the infimum in $[0,1]$ of the set $\{\varphi(a)^{\mathcal{M}} : a \in M\}$.

Where all terms $t_1,\ldots,t_n$ are $L(M)$-terms in which no variables occur.

Given an $L(M)$-formula $\varphi(x_1,\ldots,x_n)$ let $\varphi^{\mathcal{M}} : M^n \to [0,1]$ denote the function defined by
$$\varphi^{\mathcal{M}}(a_1,\ldots,a_m) = (\varphi(a_1\ldots,a_n))^{\mathcal{M}}.$$

An important fact about formulas in continuous logic is that they define uniformly continuous functions whose modulus of uniform continuity does not depend on $\mathcal{M}$ but only on the data given by the signature L. This is stated precisely in the following remark. We do not include the proof here but it is based in the fact that the composition of uniformly continuous function is uniformly continuous.

Remark 1.1.2. *Let $t(x_1,\ldots,x_n)$ be an L-term and $\varphi(x_1,\ldots,x_n)$ an L-formula. Then there exist functions Δ_t and Δ_φ from $(0,1]$ to $(0,1]$ such that for any L-structure $\mathcal{M}$, Δ_t is a modulus of uniform continuity for the function $t^{\mathcal{M}} : M^n \to M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^{\mathcal{M}} : M^n \to [0,1]$.*

Given two L-formulas $\varphi(x_1,\ldots,x_n)$ and $\psi(x_1,\ldots,x_n)$ we define the *logical distance* between them as
$$|\varphi - \psi| := \sup|\varphi^{\mathcal{M}}(a_1,\ldots,a_n) - \psi^{\mathcal{M}}(a_1,\ldots,a_n)|,$$
where the supremum is taking over all L-structures $\mathcal{M}$ and all $a_1,\ldots,a_n \in M$.

This function induces a pseudometric on the set of all formulas with free variables among $x_1, \ldots, x_n$. Two formulas are *logically equivalent* if the logical distance between them is 0.

In contrast with the classical theory, where we have a clear definition of what means for an element to satisfy a formula, here we just have the formula evaluated in the element. So we have to define some value of truth. To do so, we introduce the following concept.

An *L-condition* E is a formal expression of the form $\varphi = 0$, where φ is an L-formula. We say that E is *closed* if φ is a sentence.

Let E be the $L(M)$-condition $\varphi(x_1, \ldots, x_n) = 0$ and $a_1, \ldots, a_n \in M$, we say that E is *true* of $a_1, \ldots, a_n$ in $\mathcal{M}$, $\mathcal{M} \models E[a_1 \ldots, a_n]$, if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = 0$.

We adapt the definition of logical equivalence from formulas to conditions in a natural way:

Let E_1 be the L-condition $\varphi_1(x_1, \ldots, x_n) = 0$ and let E_2 be the L-condition $\varphi_2(x_1, \ldots, x_n) = 0$. We say that E_1 and E_2 are *logically equivalent* if for every L-structure $\mathcal{M}$ and every $a_1 \ldots, a_n \in M$ we have

$$\mathcal{M} \models E_1[a_1 \ldots, a_n] \iff \mathcal{M} \models E_2[a_1 \ldots, a_n].$$

Writing everything as a condition can be tedious, to simplify the notation we use the expression $\varphi = \psi$ as an abbreviation for the condition $|\varphi - \psi| = 0$ for φ and φ formulas. Since each number $r \in [0, 1]$ can be seen as a connective, expressions of the form $\varphi = r$ will also be regarded as a condition.

It is common to construct a metric space as the quotient of a pseudometric space or as the completion of such a quotient, and the same is true for metric structures. To do that construction, we need to consider what we will call prestructures and develop the semantics of continuous logic for them.

Let us fix a signature L for metric structures. An *L-prestructure* $\mathcal{M}_0$ based on (M_0, d_0) is a structure defined the same way as an L-structure, except that it is based on a pseudometric space. Given an L-prestructure $\mathcal{M}_0$, let (M, d) be the quotient metric space induced by (M_0, d_0) with quotient map $\pi : M_0 \to M$. We define a prestructure $\mathcal{M}$, in which the interpretations in $\mathcal{M}$ of the symbols of L are the natural interpretations induced by the prestructure $\mathcal{M}_0$ and π. Using the usual properties of uniformly continuous functions it is easy to check that these interpretations are well-defined and L is the signature of $\mathcal{M}$. We need one step more because we have required the space to be complete. We define an L-structure $\mathcal{N}$ by taking a completion of $\mathcal{M}$. This is based on a complete metric space (N, d) that is a completion of (M, d), and its additional structure is defined the natural way, induced by the prestructure $\mathcal{M}$. As before, usual properties of uniformly continuous functions guarantee that $\mathcal{N}$ is an L-structure.

As an example, one could construct the unit ball of $L^p((0, 1))$ for $p \in [1, \infty]$ starting with the prestructure consisting of all integrable functions from a measure space X to $(0, 1)$ which have norm less than 1. There, the norm induces a pseudometric. In this particular case, the resulting quotient prestructure is already complete, and hence a structure.

Also, as it is natural, we have that the interpretations of formulas and terms are not changed when taking the steps necessary to construct an L-structure from an L-prestructure.

Remark 1.1.3. *Let $\mathcal{M}_0$ be an L-prestructure with underlying pseudometric space (M_0, d_0); let $\mathcal{M}$ be is quotient L-prestructure with quotient map $\pi : M_0 \to M$, and let $\mathcal{N}$ be the L-structure that results from completing $\mathcal{M}$. Let $t(x_1, \ldots, x_n)$ be any L-term and $\varphi(x_1 \ldots, x_n)$ be any L-formula. Then:*

(1) $t^{\mathcal{N}}(\pi(a_1), \ldots, \pi(a_n)) = t^{\mathcal{M}}(\pi(a_1), \ldots, \pi(a_n)) = t^{\mathcal{M}_0}(a_1, \ldots, a_n)$ *for all elements* $a_1, \ldots, a_n \in M_0$*;*

(2) $\varphi^{\mathcal{N}}(\pi(a_1), \ldots, \pi(a_n)) = \varphi^{\mathcal{M}}(\pi(a_1), \ldots, \pi(a_n)) = \varphi^{\mathcal{M}_0}(a_1, \ldots, a_n)$ *for all elements* $a_1, \ldots, a_n \in M_0$*;*

We introduce some fundamental model theoretic concepts and their basic properties. Fix a signature L for metric structures.

A *theory* is a set of closed L-conditions.

We say that $\mathcal{M}$ is a *model* of T, $\mathcal{M} \models T$, if $\mathcal{M} \models E$ for every $E \in T$. We denote $Mod_L(T)$ to the collection of all L-structures that are models of T.

A theory is *complete* if it has the form of $Th(\mathcal{M})$, the set of all closed L-conditions satisfied by an L-structure $\mathcal{M}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures:

1. We say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

2. A function F from a subset of M into N is called an *elementary map* from $\mathcal{M}$ into $\mathcal{N}$ if for all $\varphi(x_1, \ldots, x_n) \in Form(L)$ and $a_1, \ldots, a_n \in Dom(F)$, we have

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(F(a_1), \ldots, F(a_n)).$$

An elementary map whose domain is all of M is called an *elementary embedding*. If the inclusion map from M into N is an elementary embedding, we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$ and write $\mathcal{M} \preccurlyeq \mathcal{N}$.

Remark 1.1.4. *Let $\mathcal{M}$ be an L-structure and $A \subseteq M$, if a and a' are elements of $\mathcal{M}$ that satisfy the same set of $L(A)$-conditions, then the map*

$$F : A \cup \{a\} \to A \cup \{a'\} \subseteq M$$

which is the identity over A and sends a to a' is an elementary map.

The next result gives a method to check if a substructure is an elementary substructure:

Proposition 1.1.5 (Tarski-Vaught Test). *Let $\mathcal{S}$ be any set of L-formulas which is dense in the set of all L-formulas with respect with logical distance. Let $\mathcal{M}$, $\mathcal{N}$ be L-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then, the following statements are equivalent:*

(1). $\mathcal{M} \preccurlyeq \mathcal{N}$;

(2). *For every L-formula $\varphi(x_1, \ldots, x_n, y)$ in $\mathcal{S}$ and $a_1, \ldots, a_n \in M$,*

$$\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) : b \in N\} = \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) : c \in M\}$$

Proof. (1) $\implies$ (2). Let $\varphi(x_1, \ldots, x_n, y)$ be any L-formula and let $a_1, \ldots, a_n \in M$, then from (1) we have

$$\begin{aligned}
&\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) : b \in N\} = (\inf \varphi(a_1, \ldots, a_n, y))^{\mathcal{N}} = \\
&(\inf \varphi(a_1, \ldots, a_n, y))^{\mathcal{M}} = \inf\{\varphi^{\mathcal{M}}(a_1, \ldots, a_n, c) : c \in M\} = \\
&\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) : c \in M\}.
\end{aligned}$$

(2) $\implies$ (1). Let us first prove that (2) holds for the set of all L-formulas. Let $\varphi(x_1, \ldots, x_n, y)$ be any L-formula. Given $\varepsilon > 0$, let $\psi(x_1, \ldots, x_n, y)$ be an L-formula in $\mathcal{S}$ that approximates $\varphi(x_1, \ldots, x_n, y)$ within ε in the logical distance. Let $a_1, \ldots, a_n \in M$. Then we have

$$\begin{aligned}
\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) : b \in N\} \geqslant \inf\{\psi^{\mathcal{N}}(a_1, \ldots, a_n, b) : b \in N\} - \varepsilon = \\
\inf\{\psi^{\mathcal{N}}(a_1, \ldots, a_n, c) : c \in M\} - \varepsilon \geqslant \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) : c \in M\} - 2\varepsilon.
\end{aligned}$$

Letting ε tend to 0, we obtain

$$\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) : b \in N\} \geqslant \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) : c \in M\}.$$

The equality now follows from $\mathcal{M} \subseteq \mathcal{N}$.

Now, we prove that this implies (1) by induction on the complexity of the formulas:

For atomic formulas is inmediate because $\mathcal{M} \subseteq \mathcal{N}$.

Connectives: Let $\varphi = u(\varphi_1, \ldots, \varphi_n)(\overline{x})$. Then, for all $\overline{a} \in M$,

$$(u(\varphi_1, \ldots, \varphi_n))^{\mathcal{M}}(\overline{a}) = u(\varphi_1^{\mathcal{M}}(\overline{a}), \ldots, \varphi_n^{\mathcal{M}}(\overline{a})) = u(\varphi_1^{\mathcal{N}}(\overline{a}), \ldots, \varphi_n^{\mathcal{N}}(\overline{a})) = (u(\varphi_1, \ldots, \varphi_n))^{\mathcal{N}}(\overline{a})$$

sup and inf: For the infimum case, let $\overline{a} \in M$, then

$$\begin{aligned}
&(\inf_y \varphi(\overline{x}, y))^{\mathcal{N}}(\overline{a}) = \inf\{\varphi^{\mathcal{N}}(\overline{a}, b) : b \in N\} = \inf\{\varphi^{\mathcal{N}}(\overline{a}, c) : c \in M\} = \\
&= \inf\{\varphi^{\mathcal{M}}(\overline{a}, c) : c \in M\} = (\inf_y \varphi(\overline{x}, y))^{\mathcal{M}}(\overline{a}).
\end{aligned}$$

The supremum case follows by using $\sup_y \varphi(\overline{x}, y) = 1 - \inf_y(1 - \varphi(\overline{x}, y))$. □

We discuss now the cardinality problem we remarked when describing the construction of formulas, the solution will be to take a dense countable set of connectives so that the cardinality of the set of formulas will be $\max(|L|, \omega)$, as usual. The results of this section will ensure that this works properly.

A *system of connectives* $\mathcal{C} = (C_n : n \geqslant 1)$ is a family where each C_n is a set of n-ary connectives. A system of connectives $\mathcal{C}$ is *full* if its closure under projection to the coordinates and composition is dense in the set of all connectives, with respect to the supremum distance.

Let $\mathcal{C}$ be a system of connectives, the collection of *$\mathcal{C}$-restricted formulas* is the smallest set of formulas that contains atomic formulas and is closed under the following:

1. If $u \in C_n$ and $\varphi_1, \ldots, \varphi_n$ are $\mathcal{C}$-restricted formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is a $\mathcal{C}$-restricted formula.
2. If φ is a $\mathcal{C}$-restricted formula, then $\sup_x \varphi$ and $\inf_x \varphi$ also are $\mathcal{C}$-restricted formulas.

The following theorem states the result we needed about the density of the restricted formulas when we are working with countable full sets of connectives.

Theorem 1.1.6. *If $\mathcal{C}$ is a full system of connectives, then, for any $\varepsilon > 0$ and any L-formula $\varphi(x_1, \ldots, x_n)$ there exists a $\mathcal{C}$-restricted formula $\psi(x_1, \ldots, x_n)$ such that*

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leqslant \varepsilon$$

for all L-structures $\mathcal{M}$ and all $a_1, \ldots, a_n \in M$.

Proof. We fix $\varepsilon > 0$ and proceed by induction on formulas.

Atomic formulas are included in the $\mathcal{C}$-restricted formulas so the statement is trivial.

Connectives. Let $\varphi = u(\varphi_1, \ldots, \varphi_n)$. Using the uniform continuity of u, we take $\delta > 0$ small enough so that if $d(x, y) < \delta$ then $d(u(x), u(y)) \leqslant \frac{\varepsilon}{2}$. Now we approximate each φ_i by a $\mathcal{C}$-restricted formula ψ_i which is within distance δ. Hence, we have

$$|u(\varphi_1, \ldots, \varphi_n) - u(\psi_1, \ldots, \psi_n)| \leqslant \frac{\varepsilon}{2}.$$

Now, we take $\widetilde{u} \in \mathcal{C}$ such that $|u - \widetilde{u}| \leqslant \frac{\varepsilon}{2}$. Therefore, using triangular inequality

$$|u(\varphi_1, \ldots, \varphi_n) - \widetilde{u}(\psi_1, \ldots, \psi_n)| \leqslant \varepsilon.$$

We only have left the quantifier case. As usual, we do the inf case because supremum is analogous. Let $\varphi(x) = \inf_y \psi(x, y)$. So, for each structure $\mathcal{M}$ and $a \in M^n$, $\varphi(a)^{\mathcal{M}} = \inf\{\psi^{\mathcal{M}}(a, b) : b \in M\}$. Now we approximate ψ by a $\mathcal{C}$-restricted formula $\widetilde{\psi}$ within distance ε and we get

$$\varphi(a)^{\mathcal{M}} = \inf\{\psi^{\mathcal{M}}(a, b) : b \in M\} \leqslant \inf\{\widetilde{\psi}^{\mathcal{M}}(a, b) : b \in M\} + \varepsilon$$
$$\varphi(a)^{\mathcal{M}} = \inf\{\psi^{\mathcal{M}}(a, b) : b \in M\} \geqslant \inf\{\widetilde{\psi}^{\mathcal{M}}(a, b) : b \in M\} - \varepsilon.$$

Hence, $\inf_y \widetilde{\psi}(x, y)$ is the required formula. □

In the light of the previous theorem, we present a very simple countable full set of connectives. This will be the connectives that we will mean when we just say restricted connectives or restricted formulas.

Before constructing it, we need to define a connective. Let $\dot{-} : [0,1]\times[0,1] \to [0,1]$ be the function defined by

$$x \dot{-} y = \begin{cases} x - y & \text{if } y \leqslant x \\ 0 & \text{if } x \leqslant y. \end{cases}$$

This function is obviously uniformly continuous.

Our system of connectives will be $\mathcal{C}_0 = (C_n : n \geqslant 1)$, where $C_1 = \{0, 1, \frac{x}{2}\}$, $C_2 = \{\dot{-}\}$ and $C_n = \varnothing$ for $n \geqslant 3$. Some relevant connectives we can build by composition and projection to the coordinates are:

$$\min(x,y) = x \dot{-} (x \dot{-} y),$$
$$\max(x,y) = 1 \dot{-} \min(1 \dot{-} x, 1 \dot{-} y),$$
$$\text{and every dyadic fraction } \frac{m}{2^n} \text{ in } [0,1].$$

Proposition 1.1.7. *$\mathcal{C}_0$ is a full system of connectives.*

Proof. We are going to apply the following result: 'Let X be compact, and let A be a sublattice of $C(X)$, the set of real valued continuous functions on X. Then $cl(A)$, with the supremum distance, contains every function f in $C(X)$ that can be approximated at each pair of points in X by a function from A.'

Firstly, note that since we can express maximum and minimum with our connectives $\mathcal{C}_0|_n$ ($\mathcal{C}_0$ restricted to n-ary connectives) is a sublattice of $C([0,1]^n)$, where the partial order is $f \leqslant g \iff \forall a \in [0,1]\ f(a) \leqslant g(a)$.

Let D be the set of dyadic fractions in $[0,1]$. We are going to prove that for each $x, y \in D$ with $x \neq y$, we have that $D^2 \subseteq \{(g(x), g(y)) : g \in \mathcal{C}_0|_1\}$. Fix $x, y \in D$ with $x < y$ and let $(a, b) \in D^2$. Suppose $b \leqslant a$. Take $m \in \mathbb{N}$ such that $a < m(y - x)$ and let $g : [0,1] \to [0,1]$ be defined by

$$g(t) = \max(a \dot{-} m(t \dot{-} x), b).$$

It is easy to see that $g \in \mathcal{C}_0|_1$ and that $g(x) = a$ and $g(y) = b$. If $a < b$ we can achieve the same result using $1 \dot{-} a$ and $1 \dot{-} b$ and the function $1 \dot{-} g(t)$.

Next, by the above mentioned result, we have to show that we can approximate an arbitrary connective u in two arbitrary points x, y of $[0,1]^n$. If $x \neq y$ at least one coordinate is different. Suppose $x_1 < y_1$ without loss of generality. Let $\alpha = u(x)$ and $\beta = u(y)$, we do the case $\beta \leqslant \alpha$, the case $\alpha < \beta$ is done changing the function that we will define the same way as $g(t)$ in the beginning of the proof. We take $a, b \in D$ with $b \leqslant a$ such that $d(a, \alpha) = d(b, \beta) \leqslant \frac{\varepsilon}{2}$ and $\tilde{x}_1, \tilde{y}_1 \in D$ close enough to x_1, y_1 for the function $g(t) = \max(a \dot{-} m(t \dot{-} \tilde{x}_1), b)$ to satisfy $d(g(x_1), g(\tilde{x}_1)) \leqslant \frac{\varepsilon}{2}$ and $d(g(y_1), g(\tilde{y}_1)) \leqslant \frac{\varepsilon}{2}$. Finally, let $h(x) = g(\pi_1(x))$, this function satisfies

$$d(h(x), u(x)) = d(g(x_1), \alpha) \leqslant d(g(x_1), g(\tilde{x}_1)) + d(a, \alpha) \leqslant \varepsilon$$
$$d(h(y), u(y)) = d(g(y_1), \alpha) \leqslant d(g(y_1), g(\tilde{y}_1)) + d(b, \beta) \leqslant \varepsilon$$

The case $x = y$ is trivially approximated because D is dense in $[0,1]$. Then, the result stated at the beginning of the proof ensures that $\mathcal{C}_0|_n$ is dense in the set of n-ary connectives with respect to the supremum distance, for all n, that is, $\mathcal{C}_0$ is a full system of connectives. □

We explain here an argument that will be frequently used in the remaining sections of this document.

Proposition 1.1.8. *Let $E = \{\varphi_i(x) = 0 : i \in I\}$ be a set of L-conditions, then we can assume without loss of generality that its cardinality is less than $\max(\mathrm{card}(L), \omega)$. That is, there is a subset $D \subseteq E$ of cardinality at most $\max(\mathrm{card}(L), \omega)$ such that, for any L-structure $\mathcal{M}$ and any $a \in M^n$, the element a satisfies all the conditions in E if and only if a satisfies all the conditions in D.*

Proof. We consider the set of L-formulas $\widetilde{E} = \{\varphi_i(x) : i \in I\}$. We claim that we can chose a subset $\widehat{D} \subseteq \widetilde{E}$ of cardinality at most $\max(\mathrm{card}(L), \omega)$ that is dense in $\widetilde{E}$, with respect to the logical distance. To prove this, consider the space $Form(L)$ with the logical distance, we know that the set of restricted formulas $\mathcal{C}$ is a dense subset of the required cardinality. Let $\mathscr{B}$ be the family of balls of rational radius where the centre is an element of $\mathcal{C}$ and consider $\mathscr{B}_0 = \{B \cap \widetilde{E} : B \in \mathscr{B} \text{ and } B \cap \widetilde{E} \neq \varnothing\}$. Choosing any element x_B in every set B of the family $\mathscr{B}_0$, we construct the set $\widetilde{D} = \{x_B : B \in \mathscr{B}_0\}$, which is a dense subset of $\widetilde{E}$ with respect to the logical distance and has cardinality at most $\max(\mathrm{card}(L), \omega)$. Then, all L-conditions of the set $D = \{\varphi(x) = 0 : \varphi \in \widetilde{D}\}$ are satisfied by an element a of an L-structure $\mathcal{M}$ if and only if a satisfies all the conditions in E. □

1.2. Construction of models

We start this section by discussing ultrafilter limits in topology. Let X be a topological space and let $(x_i)_{i\in I}$ be a collection of elements of X. If D is an ultrafilter on I and $x \in X$, we write

$$\lim_{i,D} x_i = x$$

and say that x is the D-limit of $(x_i)_{i\in I}$ if for every neighborhood U of x, the set $\{i \in I : x_i \in U\}$ belongs to D.

Sometimes, it can be useful to consider the collection $(x_i)_{i\in I}$ as a the image of a function $f : I \to X$, then x is the D-limit of $(x_i)_{i\in I}$ if for every neighborhood U of x, we have $f^{-1}(U) \in D$.

Lemma 1.2.1. *Let X be a topological space:*

(1) *The topology on X is compact if and only if for every collection $(x_i)_{i\in I}$ of elements of X, and every ultrafilter D on I the D-limit of $(x_i)_{i\in I}$ exists.*

(2) *The topology on X is Hausdorff if and only if for every collection $(x_i)_{i\in I}$ of elements of X, and every ultrafilter D on I the D-limit of $(x_i)_{i\in I}$, if exists, is unique.*

Proof. (1) Assume X is compact. Let $f : I \to X$ be a function and D an ultrafilter on I. Suppose $(x_i)_{i\in I}$ has no D-limit. Hence, for every $x \in X$ there exists a neighborhood U_x of x such that $f^{-1}(U_x) \notin D$. We have that $\{U_x : x \in X\}$ is a cover of X. By compactness, there exists a finite subcover $U_{x_1}, \dots, U_{x_n}$ of X, therefore $\bigcup_{i=1}^{n} f^{-1}(U_{x_i}) = I$, but this is a contradiction.

Assume now that the required limits exist. Suppose that X is not compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X with no finite subcover. Then every intersection of finitely many elements of the family $\mathcal{A} = \{U_\alpha^c : \alpha \in \Lambda\}$ is not empty. Hence, $\mathcal{A}$ can be extended to an ultrafilter D on X. Let x be the D-limit of $(i)_{i\in X}$. Take $\alpha \in \Lambda$ such that $x \in U_\alpha$, but then, by our definition of D-limit $U_\alpha \in D$. This contradicts that D is an ultrafilter because $U_\alpha^c \in D$.

(2) Assume that X is Haussdorf. We proceed by contradiction. Let $x, y \in X$ be such that $y = \lim_{i,D} x_i$ and $x = \lim_{i,D} x_i$. Let $U, V \subseteq X$ be two open sets such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. By our definition of D-limit we have that $\{i \in I : x_i \in U\} \in D$ and $\{i \in I : x_i \in V\} \in D$, but this is a contradiction because those two sets are disjoint.

Assume the uniqueness of the required limits. Suppose that X is not Hausdorff, that is, there exist $x, y \in X$, $x \neq y$, such that for all $U, V \subseteq X$ neighborhoods of x and y respectively $U \cap V \neq \varnothing$. This means that we can enlarge the set

$$\{A \subseteq X : A \text{ is neighborhood of } x \text{ or } y\}$$

to an ultrafilter D on X. Finally,

$$y = \lim_{i,D} i = x,$$

where we are taking $I = X$. □

The following lemmas show that the D-limit behaves well with respect to continuous functions, supremum and infimum.

Lemma 1.2.2. *Let X, X' be topological spaces and $F : X \to X'$ be a continuous function. For any collection $(x_i)_{i\in I}$ from X and any ultrafilter D on I we have that:*

$$\lim_{i,D} x_i = x \implies \lim_{i,D} F(x_i) = F(x)$$

where the ultrafilter limits are taken in X and X' respectively.

Proof. Let U be an open neighbourhood of $F(x)$ in X'. Due to the continuity of F, $F^{-1}(U)$ is an open neighbourhood of x. By definition of D-limit of a sequence $(x_i)_{i\in I}$, there exists $A \in D$ such that for all $i \in A$, $x_i \in F^{-1}(U)$. Hence, for all $i \in A$, $F(x_i) \in U$. □

Lemma 1.2.3. *Let X be a closed, bounded interval in $\mathbb{R}$. Let $(\mathcal{S}_i : i \in I)$ be any collection of sets and let $(F_i : i \in I)$ be a family of functions $F_i : \mathcal{S}_i \to X$. Then, for* *any ultrafilter D on I*

$$\sup_x(\lim_{i,D} F_i(x_i)) = \lim_{i,D}(\sup_{x_i} F_i(x_i))$$
$$\inf_x(\lim_{i,D} F_i(x_i)) = \lim_{i,D}(\inf_{x_i} F_i(x_i)),$$

where in the left hand side, the supremum and infimum are taken over all collections $x = (x_i)_{i\in I} \in \prod_{i\in I} \mathcal{S}_i$ and in the right hand side the supremum and the infimun are taken over each $\mathcal{S}_i$.

Proof. The sup and inf cases are analogous, so we just do the sup one.

Let $r_i = \sup_{x_i} F_i(x_i)$ for each $i \in I$ and let $r = \lim_{i,D} r_i$. Let $\varepsilon > 0$ and let $A(\varepsilon)$ be an element of D such that r_i is within distance ε from r for each $i \in A(\varepsilon)$.

We prove the equality in two parts.

For each $\varepsilon > 0$, we have $F_i(x_i) \leqslant r_i \leqslant r + \varepsilon$ for each $i \in A(\varepsilon)$. Hence, we have that $\lim_{i,D} F_i(x_i) \leqslant r + \varepsilon$. Letting ε tend to 0 and taking the supremum gives us the inequality $\sup_x(\lim_{i,D} F_i(x_i)) \leqslant \lim_{i,D}(\sup_{x_i} F_i(x_i))$.

For the other inequality, fix $\varepsilon > 0$ and for each $i \in I$ chose $x_i \in \mathcal{S}_i$ such that $r_i \leqslant F_i(x_i) + \frac{\varepsilon}{2}$. For each $i \in A(\frac{\varepsilon}{2})$ one has that $r \leqslant F_i(x_i) + \varepsilon$. Taking D-limit first and supremum after that we get $\lim_{i,D}(\sup_{x_i} F_i(x_i)) \leqslant \sup_x(\lim_{i,D} F_i(x_i)) + \varepsilon$. Letting ε tend to 0 gives us the required result. □

To construct the ultraproduct of metric structures, first we have to discuss about ultraproducts of metric spaces and functions.

Let us start by studying the structure of the product of metric spaces. Let $((M_i, d_i) : i \in I)$ be a family of bounded metric spaces, all having a common bound K for the diameter and let D be an ultrafilter on I. We define a function d on the cartesian product $\prod_{i\in i} M_i$ by

$$d(x, y) = \lim_{i,D} d_i(x_i, y_i),$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$. Note that this D-limit is taken in $[0, K]$ so the existence and uniqueness is guaranteed. It is clear that this defines a pseudometric on the cartesian product, so we can define the natural equivalence relation $x \sim_D y$ if and only if $d(x, y) = 0$. Furthermore, the pseudometric d induces a metric in the quotient space

$$(\prod_{i\in I} M_i)_D = (\prod_{i\in I} M_i)/\sim_D$$

that we also denote d. The metric space $((\prod_{i\in I} M_i)_D, d)$ is called the D-*ultraproduct* of $((M_i, d_i) : i \in I)$ and the equivalence class of $(x_i)_{i\in I} \in \prod_{i\in I} M_i$ is denoted $((x_i)_{i\in I})_D$.

Note that when we defined metric structures, we required the space to be complete, the following lemma ensures that we will not have any difficulties when constructing the ultraproduct of a family of structures.

Proposition 1.2.4. *Let $((M_i, d_i) : i \in I)$ be a family of complete, bounded metric spaces, all having diameter less than K. Let D be an ultrafilter on I and let (M, d) be the D-ultraprouct of $((M_i, d_i) : i \in I)$. The metric space (M, d) is complete.*

Proof. It suffices to show that every Cauchy sequence has a limit in M. Let $(x^k)_{k\geqslant 1}$ be a Cauchy sequence, moreover, we may assume that $d(x^k, x^{k+1}) < 2^{-k}$ holds for all $k \geqslant 1$. Let $(x_i^k)_{i\in I}$ be a representative of x^k for each $k \geqslant 1$. For each $m \geqslant 1$ let A_m be the set of all $i \in I$ such that $d_i(x_i^k, x_i^{k+1}) < 2^{-k}$ holds for all $k = 1, \ldots, m$. Obviously, the sets $(A_m)_{m\geqslant 1}$ form a decreasing chain, they are also all in D because they are finite intersections of sets that are in D by definition of the distance d. Now, we construct a representative $(y_i)_{i\in I}$ of the limit of the Cauchy sequence $(x^k)_{k\geqslant 1}$ in (M, d). If $i \notin A_1$, we take y_i to be an arbitrary element of M_i. If $i \in A_m \backslash A_{m+1}$ for some $m \geqslant 1$, we take $y_i = x_i^{m+1}$. If $i \in A_m$ for all $m \geqslant 1$, then $(x_i^k)_{k\geqslant 1}$ is a Cauchy sequence in a complete metric space (M_i, d_i), so we take y_i to be its limit.

Now, for each $m \geqslant 1$ and each $i \in A_m$, we claim that $d_i(x_i^m, y_i) \leqslant 2^{-m+1}$. To show this, we check the two cases. If $i \notin A_{m+k}$ for some $k \geqslant 1$, then $y_i = x_i^{m+k}$ and $d_i(x_i^m, y_i) < 2^{-(m+k)} + \cdots + 2^{-m} \leqslant 2^{-m+1}$. If $i \in A_k$ for all $k \geqslant 1$, then y_i is the limit of the Cauchy sequence $(x_i^k)_{k\geqslant 1}$ and we use that in a Cauchy sequence like the ones we are assuming, the distance between the m-th term and the limit is at most 2^{-m+1}. Hence, $((y_i)_{i\in I})_D$ is the limit in (M, d) of the sequence $(x^k)_{k\geqslant 1}$. □

Let us note a particular case of the ultraproduct of metric spaces.

Remark 1.2.5. *If all (M_i, d_i) are the same metric space (M, d), the construction above is called the D-ultrapower and its denoted by $(M)_D$. We can define the diagonal map $T : M \to (M)_D$ as $T(x) = ((x)_{i\in I})_D$. The diagonal map is an isometric embedding, moreover, if (M, d) is a compact metric space, then T is also surjective.*

Proof. Take any $(x_i)_{i\in I} \in \prod_{i\in I} M_i$. As M is compact, by lemma 1.2.1, this sequence has a D-limit x. Then, by definition of the metric in $(\mathcal{M})_D$ and properties of the D-limit, we have $d((x_i)_{i\in I}, T(x)) = 0$, so $T(x) = ((x_i)_{i\in I})_D$. □

Now, we define ultraproduct of functions.

Let $((M_i, d_i) : i \in I)$ and $((M_i', d_i') : i \in I)$ be families of metric spaces all of them with diameter less than K. Fix $n \geqslant 1$ and let $f_i : M_i^n \to M_i'$ be a uniformly continuous function for each $i \in I$, all of them with the same modulus of uniform continuity Δ. Then, given an ultrafilter D on I, we define a function

$$(\prod_{i\in I} f_i)_D : (\prod_{i\in I} M_i)_D^n \to (\prod_{i\in I} M_i')_D$$

by setting

$$(\prod_{i\in I} f_i)_D(((x_i^1)_{i\in I})_D, \ldots, ((x_i^n)_{i\in I})_D) = ((f_i(x_i^1, \ldots, x_i^n))_{i\in I})_D$$

for all $(x_i)_{i\in I} \in \prod_{i\in I} M_i$.

As in the ultraproduct of metric spaces, we note here that we required functions of our metric structures to be uniformly continuous. The next lemma ensures that we will have no problem when defining the ultraproduct of metric structures.

Lemma 1.2.6. *The function defined above is well defined and uniformly continuous with modulus of uniform continuity* Δ.

Proof. For simplicity, we do the case $n = 1$, the case $n > 1$ is analogous because we use the maximum distance when we discuss finite powers of a metric space.

Fix $\varepsilon > 0$ and let $((x_i)_{i\in I})_D, ((y_i)_{i\in I})_D \in (\prod_{i\in I} M_i)_D$ be a pair of points such that $d(((x_i)_{i\in I})_D, ((y_i)_{i\in I})_D) < \Delta(\varepsilon)$. Then, by definition of the distance and the D-limit, there exist $A \in D$ such that $d_i(x_i, y_i) < \Delta(\varepsilon)$ for all $i \in A$. Since Δ is a modulus of uniform continuity for all of the functions f_i, we have that $d_i(f_i(x_i), f_i(y_i)) \leqslant \varepsilon$ for all $i \in A$. Hence $d((f_i((x_i))_{i\in I})_D, (f_i((y_i))_{i\in I})_D) \leqslant \varepsilon$. This shows that $(\prod_{i\in I} f_i)_D$ is well defined and has Δ as a modulus of uniform continuity. □

Note that in the latter proof it is the first time that our precise definition of uniform continuity is relevant.

Finally, we define the ultraproduct of metric structures. Let $(\mathcal{M}_i : i \in I)$ be a family of L-structures with underlying metric spaces (M_i, d_i) and D an ultrafilter on I. As all $\mathcal{M}_i$ are L-structures and L includes D_L, a bound for the diameter of the L-structures, there exist an uniform bound on the diameters of all M_i, so we may form their D-ultraproduct. Moreover, for each function or predicate symbol in L, as their modulus of uniform continuity are included in L, their interpretations have the same modulus of uniform continuity, so their D-ultraproduct is well defined. In the case of predicates, we identify the D-ultrapower of $[0, 1]$ with $[0, 1]$ itself (see remark 1.2.5).

Therefore, we can define the D-ultraproduct of the family $(\mathcal{M}_i : i \in I)$, usually denoted by $(\prod_{i\in I} \mathcal{M}_i)_D$, as the following L-structure

$$(\prod_{i\in I} \mathcal{M}_i)_D = ((\prod_{i\in I} M_i)_D, (\prod_{i\in I} P_k^{\mathcal{M}_i})_D, (\prod_{i\in I} f_j^{\mathcal{M}_i})_D, ((a_l^{\mathcal{M}_i})_{i\in I})_D : j \in J, k \in K, l \in L),$$

Where K, J and L are the set of index of functions, predicates and constants in L.

Next, we have the analogous result in continuous model theory to Łoś's fundamental theorem of ultraproducts.

Theorem 1.2.7. *Let* $(\mathcal{M}_i : i \in I)$ *be a family of* L*-structures. Let* D *be any ultrafilter on* I *and let* $\mathcal{M}$ *be the* D*-ultraproduct of* $(\mathcal{M}_i : i \in I)$. *Let* $\varphi(x_1, \ldots, x_n)$ *be an* L*-formula. If* $a_k = ((a_i^k)_{i\in I})_D$ *are elements of* $\mathcal{M}$ *for* $k = 1, \ldots, n$, *then*

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \lim_{i,D} \varphi^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n).$$

Proof. We proceed by induction of formulas.

Atomic formulas. Let P be a predicate symbol in L and let $((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D$ be elements of M. Then, for any $b \in [0, 1]$, using the diagonal embedding to identify

b as $(b)_{i\in I} \in \prod_{i\in I}[0,1]$ we have that $P^{\mathcal{M}}(((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D) = b$ if and only if $d(\prod_{i\in I} P^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n), (b)_{i\in I}) = 0$. This is equivalent, by our definition of limit, to $\lim_{i,D} P^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = b$.

Connectives. Let u be an n-ary connective,let $\varphi_1, \ldots, \varphi_n$ be L-formulas and let $((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D$ be elements of M. Using the induction hypothesis and lemma 1.2.2 we get the following equalities.

$$\begin{aligned}
&(u(\varphi_1, \ldots, \varphi_n))^{\mathcal{M}}((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D = \\
&u(\lim_{i,D} \varphi_1(a_i^1, \ldots, a_i^n), \ldots, \lim_{i,D} \varphi_n(a_i^1, \ldots, a_i^n)) = \\
&\lim_{i,D}(u(\varphi_1^{\mathcal{M}_i}, \ldots, \varphi_n^{\mathcal{M}_i})(a_i^1, \ldots, a_i^n)) \\
&\lim_{i,D}(u(\varphi_1, \ldots, \varphi_n)^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)).
\end{aligned}$$

Quantifiers. Let $\varphi(x)$ be $\inf_y \psi(x,y)$ and let $((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D$ be elements of M. We have the following equalities by lemma 1.2.3.

$$\begin{aligned}
&(\inf_y \psi(x,y))^{\mathcal{M}}(((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D) = \\
&\inf\{\psi^{\mathcal{M}}(((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D, ((b_i)_{i\in I})_D) : ((b_i)_{i\in I})_D \in M\} = \\
&\inf\{\lim_{i,D} \psi^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n, b_i) : b_i \in M_i, i \in I\} = \\
&\lim_{i,D}(\inf\{\psi^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n, b_i) : b_i \in M_i\}) = \lim_{i,D}((\inf_y \psi(x,y))^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)).
\end{aligned}$$

Hence,

$$(\varphi(((a_i^1)_{i\in I})_D, \ldots, ((a_i^n)_{i\in I})_D))^{\mathcal{M}} = \lim_{i,D}((\varphi(a_i^1, \ldots, a_i^n))^{\mathcal{M}_i}).$$

Which is the required result.

□

Corollary 1.2.8. *If $\mathcal{M}$ is an L-structure and $T : M \to (M)_D$ is the diagonal embedding, then T is an elementary embedding of $\mathcal{M}$ into $(\mathcal{M})_D$.*

Proof. Let φ be an L-formula and $a_1, \ldots, a_n \in M$. Then, by the previous result

$$\varphi^{(\mathcal{M})_D}(T(a_1), \ldots, T(a_n)) = \lim_{i,D} \varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{M}}(a_1, \ldots, a_n).$$

As the domain of T is all M, T is an elementary embedding. □

Corollary 1.2.9. *If $\mathcal{M}$ and $\mathcal{N}$ are L-structures with isomorphic ultrapowers, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $E \in Th(\mathcal{M})$ be the closed L-condition $\varphi = 0$. By the previous result, we have that $\varphi^{\mathcal{M}} = 0$ implies $\varphi^{(\mathcal{M})_D} = 0$. By the isomorphism between ultrapowers we get $\varphi^{(\mathcal{N})_D} = 0$, and, as φ has no free variables, that means $\varphi^{\mathcal{N}} = 0$ and we conclude $E \in Th(\mathcal{N})$. □

The converse to the preceding result is also true. It is the analogous to the Keisler-Shelah theorem in continuous model theory. The proof of this results if out of the scope of this memoir.

Theorem 1.2.10. [5, theorem 5.7] *If $\mathcal{M}$ and $\mathcal{N}$ are L-structures such that $\mathcal{M} \equiv \mathcal{N}$, then there exist an ultrafilter D such that $(\mathcal{M})_D \cong (\mathcal{N})_D$.*

The following result is the analogous to the compactness theorem in continuous model theory.

Theorem 1.2.11 (Compactness theorem)**.** *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T is finitely satisfiable in $\mathcal{C}$. Then, there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

Proof. Let Λ be $\mathcal{P}^{<\omega}(T)$. For each $\lambda \in \Lambda$, let $\mathcal{M}_\lambda$ denote a fixed structure in $\mathcal{C}$ such that $\mathcal{M}_\lambda \models E$, for each $E \in \lambda$, which exists by hypothesis.

For each L-condition $E \in T$, let $S(E) = \{\lambda \in \Lambda : E \in \lambda\}$. As the collection $\{S(E) : E \in T\}$ has the finite intersection property, there exists an ultrafilter D on Λ such that D contains all $S(E)$ for $E \in T$.

Now, let $\mathcal{M} = (\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda)_D$. Note that for every $E \in T$ and $\lambda \in S(E)$, $\mathcal{M}_\lambda \models E$. So the set $\{\lambda \in \Lambda : \mathcal{M}_\lambda \models E\}$ belongs to D. Hence, for every L-condition $E := (\varphi = 0)$ we can apply theorem 1.2.7 to φ and we get $\varphi^{\mathcal{M}} = \lim_{\lambda,D} \varphi^{\mathcal{M}_\lambda}$. Since for every $\lambda \in S(E)$, $\varphi^{\mathcal{M}_\lambda} = 0$, properties of the limit imply $\lim_{\lambda,D} \varphi^{\mathcal{M}_\lambda} = 0$ and hence $\mathcal{M} \models E$. As E was arbitrary, $\mathcal{M} \models T$. □

In the context of the continuous model theory, the compactness theorem gives a better result that in the classical setting. To explore this improvement we introduce the following definition.

For any set Σ of L-conditions, Σ^+ is the set of all conditions $\varphi \leqslant \frac{1}{n}$ such that $\varphi = 0$ is an element Σ and $n \geqslant 1$.

Corollary 1.2.12. *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T^+ is finitely satisfiable in $\mathcal{C}$. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

Proof. Applying the compactness theorem 1.2.11 to T^+, we get an L-structure $\mathcal{M}$ which is an ultraproduct of structures of $\mathcal{C}$ and a model of T^+. Note that every L-structure which is a model of T^+ is also a model of T. Hence, we have the required result. □

Let T be an L-theory and $\Sigma(x_j : j \in J)$ a set of L-conditions. We say that Σ is *consistent* with T if for every finite subset F of Σ there exist $\mathcal{M} \in Mod_L(T)$ and a tuple a of elements in M such that for every condition $E \in F$, $\mathcal{M} \models E[a]$.

Corollary 1.2.13. *Let T be an L-theory and $\Sigma(x_i : i \in I)$ a set of L-conditions, and assume that Σ^+ is consistent with T. Then there exists $\mathcal{M} \in Mod_L(T)$ and a set of elements $\{a_i : i \in I\}$ of M such that*

$$\mathcal{M} \models E[a_i : i \in I]$$

for every L-condition $E \in \Sigma$.

Proof. We introduce a new set of constants $\{c_i : i \in I\}$ ans we consider the extended signature $L(\{c_i : i \in I\})$. The consistency of Σ^+ implies that the $L(\{c_i : i \in I\})$-theory $T \cup \Sigma^+(c_i : i \in I)$ is finitely satisfiable in the class of $L(\{c_i : i \in I\})$-structures $\mathcal{C} = \{(\mathcal{M}, (c_i : i \in I)) : \mathcal{M} \models T\}$, that is, all models of T and all possible assignation of the constants to elements of $\mathcal{M}$. Applying compactness theorem to the $L(\{c_i : i \in I\})$ theory $T \cup \Sigma^+(c_i : i \in I)$ and the class $\mathcal{C}$ of $L(\{c_i : i \in I\})$-structures yields a model $\mathcal{M}$ of T where the interpretations of c_i for $i \in I$ satisfy $\Sigma^+(x_i : i \in I)$, hence they satisfy $\Sigma(x_i : i \in I)$. □

Proposition 1.2.14. *Suppose that $\mathcal{C}$ is a clsss of L-structures. The following statements are equivalent:*

(1) *$\mathcal{C}$ is axiomatizable in L.*

(2) *$\mathcal{C}$ is closed under isomorphisms and ultraproducts, and its complement is closed under ultrapowers.*

Proof. (1) $\implies$ (2). If $\mathcal{C}$ is axiomatizable, then is closed under isomorphisms because isomorphic models have the same theory. The same is true for ultraproducts because theorem 1.2.7 implies that if all structures of $\mathcal{C}$ involved in the ultraproduct satisfy the L-condition $\varphi = 0$, then the ultraproduct also satisfy $\varphi = 0$. The complement of $\mathcal{C}$ is closed under ultrapowers by corollary 1.2.8.

(2) $\implies$ (1).Let T be the set of closed L-conditions satisfied by every structure in $\mathcal{C}$. We are going to prove that $\mathcal{M} \models T$ if and only if $\mathcal{M} \in \mathcal{C}$. Assume first that $\mathcal{M} \models T$, we claim that $Th(\mathcal{M})^+$ is finitely satisfiable in every L-structure of $\mathcal{C}$. To prove the claim, supose that it is false, that is, there exists sentences $\varphi_1, \ldots, \varphi_n$ and $\varepsilon > 0$ such that $\varphi_i^{\mathcal{M}} = 0$ for all $i = 1, \ldots, n$ but for any $\mathcal{N} \in \mathcal{C}$ we have that $\varphi_j^{\mathcal{N}} \geqslant \varepsilon$ for some $j = 1, \ldots, n$. This gives us a contradiction because $\max(\varphi_1, \ldots, \varphi_n) \geqslant \varepsilon$ is an L-condition in T but is not satisfied in $\mathcal{M}$.

As $Th(\mathcal{M})^+$ is finitely satisfiable in $\mathcal{C}$, compactness theorem 1.2.11 implies the existence of a model $\mathcal{M}'$ of $Th(\mathcal{M})^+$ which is an ultraproduc of structures in $\mathcal{C}$ and hence $\mathcal{M}' \in \mathcal{C}$. Since every model of $Th(\mathcal{M})^+$ is a model of $Th(\mathcal{M})$, this implies $\mathcal{M}' \equiv \mathcal{M}$. Finally, theorem 1.2.10 implies that there exists an ultrafilter D such that $(\mathcal{M}')_D \cong (\mathcal{M})_D$ and our assumptions of $\mathcal{C}$ imply $\mathcal{M} \in \mathcal{C}$. □

Let Λ be a linearly ordered set, a chain of L-structures is a family of L-structures $(\mathcal{M}_\lambda : \lambda \in \Lambda)$ such that $\mathcal{M}_\lambda \subseteq \mathcal{M}_\eta$ for $\lambda < \eta$. If we have such a family, we can define its union as an L-prestructure in the natural way. Note that we say prestructure because an arbitrary union of complete metric spaces may not be complete. After taking the completion, we denote the resulting L-structure by $\bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda$ and refer to it as the *union of the chain of L-structures.*

Remark 1.2.15. *If the cofinality of Λ is not countable, then the union $\bigcup_{\lambda\in\Lambda} M_\lambda$ of the metric spaces is complete. Hence, the universe of the L-structure $\bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is $\bigcup_{\lambda\in\Lambda} M_\lambda$.*

Proof. Let a be an element of the completion of $\bigcup_{\lambda\in\Lambda} M_\lambda$. There exist a Cauchy sequence $(a_n)_{n\geqslant 1}$ in $\bigcup_{\lambda\in\Lambda} M_\lambda$ converging to a. The cofinality of Λ implies the existence of $\alpha < \lambda$ such that $a_n \in M_\alpha$ for all $n \geqslant 1$. Hence, as M_α is complete, $a \in M_\alpha \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda$. □

A chain of structures $(\mathcal{M}_\lambda : \lambda \in \Lambda)$ is called an *elementary chain* if $\mathcal{M}_\lambda \preccurlyeq \mathcal{M}_\eta$ for all $\lambda < \eta$.

The next result is the analogous to the Tarski chain lemma in the classical theory.

Proposition 1.2.16. *If $(\mathcal{M}_\lambda : \lambda \in \Lambda)$ is an elementary chain and $\lambda \in \Lambda$, then we have $\mathcal{M}_\lambda \preccurlyeq \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda$.*

Proof. By construction, we already know $\mathcal{M}_\lambda \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda = \mathcal{M}$. Now, we apply the Tarski-Vaught theorem 1.1.5. We have to check that for any L-formula $\varphi(x, y)$ and any $a \in M_\lambda^n$ with $n \geqslant 1$

$$\inf\{\varphi^{\mathcal{M}}(a, b) : b \in M\} = \inf\{\varphi^{\mathcal{M}}(a, c) : c \in M_\lambda\}.$$

Obviously, the left hand side is at most equal to the right hand side. Suppose that is strictly less, then, as $\bigcup_{\lambda\in\Lambda} M_\lambda$ is dense in M, there exists $\widetilde{b} \in \bigcup_{\lambda\in\Lambda} M_\lambda$ such that $\varphi^{\mathcal{M}}(a, \widetilde{b}) < \inf\{\varphi^{\mathcal{M}}(a, c) : c \in M_\lambda\}$. However $\widetilde{b} \in M_\beta$ for some $\beta \geqslant \lambda$, so the previous inequality contradicts $\mathcal{M}_\lambda \preccurlyeq \mathcal{M}_\beta$. Hence, we have the equality between the two sides and Tarski Vaught yields $\mathcal{M}_\lambda \preccurlyeq \mathcal{M}$. □

The *density character* of a topological space X is the least cardinal among $card(A)$ for $A \subseteq X$ dense subset. We denote this cardinal by $density(X)$

Proposition 1.2.17 (Lowenheim-Skolem)**.** *Let κ be an infinite cardinal and assume $card(L) \leqslant \kappa$. Let $\mathcal{M}$ be an L-structure and suppose $A \subseteq M$ satisfies $density(A) \leqslant \kappa$. Then there exist a substructure $\mathcal{N}$ of $\mathcal{M}$ such that*

1. *$\mathcal{N} \preccurlyeq \mathcal{M}$;*

2. *$A \subseteq N$;*

3. *$density(N) \leqslant \kappa$.*

Proof. Let A_0 be a dense subset of A of cardinality at most κ. We first prove that we can enlarge A_0 to obtain a prestructure with universe N_0, whose metric is induced by the one on M, such that $A_0 \subseteq N_0 \subseteq M$, $card(N_0) \leqslant \kappa$ and which has the following closure property: for every restricted L-formula $\varphi(x_1, \ldots, x_n, x_{n+1})$ and every rational $\epsilon > 0$, if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n, c) \leqslant \epsilon$ with $a_1, \ldots, a_n \in N_0$ and $c \in M$, then there exist $b \in N_0$ such that $\varphi^{\mathcal{M}}(a_1, \ldots, a_n, b) \leqslant \epsilon$. We start with A_0 and we enlarge it to satisfy the above closure property except that we restrict to the case $a_1, \ldots, a_n \in A_0$ and $c \in M$, then, we add the images of all tuples of elements of this enlarged set by functions in L. We call the new set A_1, as the cardinality of A_0, the cardinality of the set of restricted formulas and the cardinality of L are all at most κ. The cardinality of A_1 is at most κ. Repeating this process we get $A_0, A_1, \ldots$ and increasing chain of sets, then $N_0 = \bigcup_{k \in \mathbb{N}} A_k$ has the required closure property. The set N_0 is the universe of an L-prestructure since its a metric space (with the metric induced by the metric on M) and its closed by constants and functions in L.

Let N be the topological closure of N_0 in M, since $A_0 \subseteq N_0$, $A \subseteq N$. Since N_0 has cardinality less than κ, $density(N) \leqslant \kappa$. Using that a closed subspace of a complete metric space is also complete, N is the universe of an L-structure $\mathcal{N}$. To prove $\mathcal{N} \preccurlyeq \mathcal{M}$, we first prove that $\mathcal{N} \subseteq \mathcal{M}$, then we apply Tarski-Vaught 1.1.5. Let $\bar{c} \in L$ be a constant symbol and let $\varphi(x)$ be $d(\bar{c}, x)$. Since $\mathcal{M}$ is an L-structure, there exists $c \in M$ such that $\varphi^{\mathcal{M}}(c) = 0$. Thus, by construction of N_0, for each $n \in \omega$, $n \neq 0$ there exists $c_n \in N_0$ such that $\varphi^{\mathcal{M}}(c_n) \leqslant \frac{1}{n}$ so c belongs to the closure of N_0. Now, let f be an m-ary function symbol of L and let $\varphi(x_1, \ldots, x_m, y)$ be $d(f(x_1, \ldots, x_m), y)$. Let $a_1, \ldots, a_m \in N_0$, since $\mathcal{M}$ is an L-structure and $N_0 \subseteq M$, there exists $b \in M$ such that $\varphi^{\mathcal{M}}(a_1, \ldots, a_m, b) = 0$. Thus, by construction of N_0, for each $n \in \omega$, $n \neq 0$ there exists $b_n \in N_0$ such that $\varphi^{\mathcal{M}}(a_1, \ldots, a_m, b_n) \leqslant \frac{1}{n}$, so $f^{\mathcal{M}}(a_1, \ldots, a_n)$ belongs to the closure of N_0 and since $a_1, \ldots a_n$ were arbitrary, N_0 is dense in N and f is uniformly continuous, we have that N is closed under functions. Now, we apply the Tarski-Vaught test. We need to check that for all $\varphi(x_1, \ldots, x_n)$ restricted L-formula and $a_1 \ldots, a_n \in N$

$$\inf\{\varphi^{\mathcal{M}}(a_1, \ldots, a_m, b) : b \in M\} = \inf\{\varphi^{\mathcal{M}}(a_1, \ldots, a_m, c) : c \in N\}.$$

Since $N \subseteq M$, one of the inequalities is trivial. For the nontrivial one, let $\inf\{\varphi^{\mathcal{M}}(a_1, \ldots, a_m, b) : b \in M\} = x$, for all $\delta > 0$ there exists $\widetilde{b} \in M$ such that $\varphi^{\mathcal{M}}(a_1, \ldots, a_m, \widetilde{b}) \leqslant x + \delta$, by density of N_0 and uniform continuity of the formulas, for all $\varepsilon > 0$, there exist $\widetilde{a}_1, \ldots, \widetilde{a}_n \in N_0$ such that $\varphi^{\mathcal{M}}(\widetilde{a}_1, \ldots, \widetilde{a}_m, \widetilde{b}) \leqslant x + \delta + \varepsilon$, now, we use the closure properties of N_0 and get that there exists $c \in N_0$ such that $\varphi^{\mathcal{M}}(\widetilde{a}_1, \ldots, \widetilde{a}_m, c) \leqslant x + \delta + \varepsilon$ and so $\varphi^{\mathcal{M}}(a_1, \ldots, a_m, c) \leqslant x + \delta + 2\varepsilon$. Finally, letting ε and δ go to 0, we get $\inf\{\varphi^{\mathcal{M}}(a_1, \ldots, a_m, c) : c \in N\} \leqslant x$. □

Let $\Gamma(x_1, \ldots, x_n)$ be a set of L-conditions and let $\mathcal{M}$ be an L-structure. We say that $\Gamma(x_1, \ldots, x_n)$ is *satisfiable* in $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of M such that $\mathcal{M} \models \Gamma[a_1, \ldots, a_n]$.

Let $\mathcal{M}$ be an L-structure and let κ be an infinite cardinal. We say that $\mathcal{M}$ is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$ and for all $\Gamma(x_1, \ldots, x_n)$ set of $L(A)$-

conditions, if every finite subset of Γ is satisfiable in $(\mathcal{M}, a)_{a\in A}$, then the entire set Γ is satisfiable in $(\mathcal{M}, a)_{a\in A}$.

The following result shows that in saturated models we have more similarities to the classical setting, we can analyse L-conditions using the quantifiers $\forall$ and $\exists$.

Proposition 1.2.18. *Let $\mathcal{M}$ be an L-structure and suppose $E(x_1, \dots, x_m)$ is the L-condition*

$$Q^1_{y_1} \dots Q^n_{y_n} \varphi(x_1, \dots, x_m, y_1, \dots, y_n) = 0$$

where each Q^i is either inf *or* sup *and φ is quantifier free.*

Let $\mathcal{E}(x_1, \dots, x_m)$ be the mathematical statement

$$\widetilde{Q}^1_{y_1} \dots \widetilde{Q}^n_{y_n} \varphi(x_1, \dots, x_m, y_1, \dots, y_n) = 0$$

where $\widetilde{Q}^i_{y_i}$ is $\exists y_i$ if $Q^i_{y_i}$ is $\inf_{y_i}$ and $\forall y_i$ if $Q^i_{y_i}$ is $\sup_{y_i}$.

If $\mathcal{M}$ is ω-saturated, then for any elements $a_1, \dots, a_m \in M$, we have

$$\mathcal{M} \models E[a_1, \dots, a_m] \text{ if and only if } \mathcal{E}(a_1, \dots, a_m) \text{ is true in } \mathcal{M}.$$

Proof. We proceed by induction on the number of quantifiers. The case $n = 0$ is trivial. For the case $n+1$, suppose we are considering the condition $\inf_y \psi(x_1, \dots, x_m, y) = 0$ where ψ is an L-formula with n quantifiers. Let $a_1 \dots, a_m \in M$. If there exists $b \in M$ such that $\psi(a_1, \dots, a_m, b) = 0$ it follows that $\inf_y \psi(a_1, \dots, a_m, y) = 0$. For the converse, consider $\Gamma(y) = \{\psi(a_1, \dots, a_m, y) \leqslant \frac{1}{n} : n \in \omega; n \neq 0\}$ set of $L(a_1, \dots, a_m)$-conditions, it is clear that every finite subset is satisfiable in $(\mathcal{M}, a_1 \dots, a_m)$, so by saturation there exists $b \in M$ such that $(\mathcal{M}, a_1 \dots, a_m) \models (\psi(a_1, \dots, a_m, y) = 0)[b]$. Finally we apply the induction hypothesis to the condition $\psi(a_1, \dots, a_m, b) = 0$. The $\sup_y \psi(x_1, \dots, x_m, y) = 0$ case is trivial. □

Let $\mathcal{M}$ be an L-structure and $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. We say that $\mathcal{N}$ is an *enlargement* of $\mathcal{M}$ if for every $A \subseteq M$ and $\Gamma(x_1, \dots, x_n)$ set of $L(A)$-conditions, whenever every finite subset of $\Gamma(x_1, \dots, x_n)$ is satisfiable in $(\mathcal{M}, a)_{a\in A}$, then the entire set Γ is satisfiable in $(\mathcal{N}, a)_{a\in A}$.

Lemma 1.2.19. *Every L-structure has an enlargement.*

Proof. Let $\mathcal{M}$ be and L-structure and let J be a set with cardinality bigger than the cardinality of $L(M)$-formulas (by remark 1.1.8, $\max(\omega, |L(M)|)$ suffices). Let $I = \mathcal{P}^{<\omega}(J)$ and let D be an ultrafilter on I containing the sets $S_j = \{i \in I : j \in i\}$, for $j \in J$. We claim that $\mathcal{N} = (\mathcal{M})_D$ is an enlargement of $\mathcal{M}$.

First, we already know that $\mathcal{M} \preccurlyeq \mathcal{N}$ by corollary 1.2.8. Let $A \subseteq M$ and suppose $\Gamma(x_1, \dots, x_n)$ is a set of $L(A)$-conditions such that every finite subset of Γ is satisfiable in $(\mathcal{M}, a)_{a\in A}$. Let α be a function from J onto Γ. Given $i = \{j_1, \dots, j_m\} \in I$, let $(a^1_i, \dots, a^n_i)$ be any n-tuple from M that satisfies $\{\alpha(j_1), \dots, \alpha(j_m)\}$ in $(\mathcal{M}, a)_{a\in A}$. For each $k = 1, \dots, n$ set $a_k = ((a^k_i)_{i\in I})_D$. We check that Łoś theorem 1.2.7 yields that

$(a_1, \ldots, a_n)$ satisfies Γ in $(\mathcal{N}, a)_{a\in A}$. Let $\varphi = 0$ be any condition in Γ, by Łoś theorem, $\varphi^{\mathcal{N}}(a_1, \ldots, a_m) = \lim_{i,D} \varphi^{\mathcal{M}}(a_i^1, \ldots, a_i^n)$, this limit is 0 since for any j in the preimage by α of the condition $\varphi = 0$ the set S_j is in the ultrafilter D.

□

Proposition 1.2.20. *Let $\mathcal{M}$ be an L-structure. For every infinite cardinal κ, $\mathcal{M}$ has a κ-saturated elementary extension.*

Proof. By increasing κ if necessary we may assume κ is regular and $\omega < \kappa$. By induction we construct an elementary chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ such that $\mathcal{M}_0 = \mathcal{M}$, if $\beta = \alpha + 1$ we take $\mathcal{M}_\beta$ an enlargement of $\mathcal{M}_\alpha$, if β is a limit ordinal we take $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$. Let $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$. By the Tarski chain lemma 1.2.16, $\mathcal{M}_\alpha \preccurlyeq \mathcal{N}$ for all $\alpha < \kappa$. It remains to prove that $\mathcal{N}$ is κ-saturated. To do so, let $A \subseteq N$ be a subset of cardinality strictly less than κ. Since κ is regular, remark 1.2.15 yields that the universe of $\mathcal{N}$ is $\bigcup_{\alpha<\kappa} M_\alpha$. Hence, there exists $\alpha < \kappa$ such that $A \subseteq M_\alpha$. Therefore, $\mathcal{M}_{\alpha+1}$ witness that $(\mathcal{N}, a)_{a\in A}$ realizes every finitely satisfiable set of $L(A)$-conditions. □

Let $\mathcal{M}$ be an L-structure and let κ be an infinite cardinal. We say that $\mathcal{M}$ is *κ-homogeneous* if for every elementary map $F : A \to M$, where $A \subseteq M$ satisfies $|A| < \kappa$, and every element $a \in M$, there exists an elementary map $\widetilde{F} : A \cup \{a\} \to M$ extending F.

Theorem 1.2.21. *Let $\mathcal{M}$ be an L-structure. If $\mathcal{M}$ is κ saturated, then $\mathcal{M}$ is κ-homogeneous.*

Proof. Let $A \subseteq M$ be any subset of the universe and let $F : A \to M$ be any elementary map. Let $b \in M$ be any element. Let $\Gamma(x)$ be the set of all $L(A)$-conditions satisfied by b. We define $\widetilde{\Gamma}(x)$ as the set of $L(F(A))$-conditions resulting by substituting, in each condition in $\Gamma(x)$, each apparition of $a \in A$ by $F(a)$. Using lemma 1.2.18, the set of conditions $\widetilde{\Gamma}(x)$ is finitely satisfiable. Hence, by the saturation of $\mathcal{M}$, there exists $b' \in M$ such that $\mathcal{M} \models \widetilde{\Gamma}[b']$. This implies that for any $L(A)$-formula φ, we have $\varphi^{\mathcal{M}}(b) = \varphi^{\mathcal{M}}(b')$. Furthermore, this yields that $F \cup \{(b, b')\}$ is an elementary map. □

Let $\mathcal{M}$ be an L-structure and let κ be an infinite cardinal. We say that $\mathcal{M}$ is *strongly κ-homogeneous* if for every extension $L(C)$ of L by constants with $card(C) < \kappa$ and maps $f, g : C \to M$ such that

$$(\mathcal{M}, f(c))_{c\in C} \equiv (\mathcal{M}, g(c))_{c\in C}$$

one has

$$(\mathcal{M}, f(c))_{c\in C} \cong (\mathcal{M}, g(c))_{c\in C}.$$

The next result is an auxiliary lemma that we will need later.

Lemma 1.2.22. *Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures such that $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{N}$ is τ-saturated, where τ is a cardinal bigger than $card(L)$ and $card(M)$. Let C be a set of less than τ new constants and let f, g be maps from C into M such that*

$$(\mathcal{M}, f(c))_{c\in C} \equiv (\mathcal{M}, g(c))_{c\in C}.$$

Then, there exist an elementary embedding T from $\mathcal{M}$ into $\mathcal{N}$ such that $T(f(c)) = g(c)$ for every $c \in C$.

Proof. Note that as $\mathcal{M} \preccurlyeq \mathcal{N}$, the map $h : \{f(c) : c \in C\} \to \{g(c) : c \in C\}$ sending $f(c)$ to $g(c)$ for every $c \in C$ is an elementary map from $\mathcal{M}$ to $\mathcal{N}$. Now, we fix an enumeration $(a_\alpha)_{\alpha<\gamma}$ of M where $\gamma = card(M)$ and use the τ-saturation of $\mathcal{N}$ to extend our original elementary map h. We proceed by transfinite induction. Let $A_\alpha = \{a_\beta : \beta < \alpha\}$ and assume we have already constructed a compatible family of elementary maps $f_\beta : A_\beta \to N$ for $\beta < \alpha$. If α is a limit ordinal $\bigcup_{\beta<\alpha} f_\beta$ is an elementary map from A_α to N. If $\alpha = \eta + 1$, we consider the set $\Gamma(x)$ of all $L(A_\eta)$-conditions satisfied by a_η. Using lemma 1.2.18, the set $\widetilde{\Gamma}(x)$ of $L(f_\eta(A_\eta))$-conditions resulting by substituting each apparition of $a \in A_\eta$ in Γ by $f_\eta(a)$ is finitely satisfiable, and by our hypothesis of saturation of $\mathcal{N}$ and the cardinality of all the sets, we get that there exists $b \in N$ such that $f_\eta \cup \{(a_\eta, b)\}$ is an elementary map that extends f_η. Finally, $\bigcup_{\alpha<\gamma} f_\alpha$ is the required elementary embedding. □

Theorem 1.2.23. *Let $\mathcal{M}$ be an L-structure. For every infinite cardinal κ, $\mathcal{M}$ has a κ-saturated elementary extension $\mathcal{N}$ such that each reduct of $\mathcal{N}$ to a sublanguage of L is strongly κ-homogeneous.*

Proof. We may assume that κ is regular without loss of generality. Given any L-structure $\mathcal{M}$, we construct an elementary chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ whose union has the desired properties. Let $\mathcal{M}_0 = \mathcal{M}$, for each $\alpha < \kappa$, if $\beta = \alpha + 1$ let $\mathcal{M}_\beta$ be an elementary extension of $\mathcal{M}_\alpha$ that is τ_α-saturated, where τ_α is a cardinal bigger than $card(L)$ and bigger than the cardinality of $\mathcal{M}_\alpha$; if β is a limit ordinal we take unions. Let $\mathcal{N}$ be the union of $(\mathcal{M}_\alpha : \alpha < \kappa)$. Tarski chain lemma 1.2.16 yields $\mathcal{M} \preccurlyeq \mathcal{N}$. To prove the κ regularity of $\mathcal{N}$, we argument as in proposition 1.2.20.

Assume that $(\mathcal{N}, f(c))_{c\in C} \equiv (\mathcal{N}, g(c))_{c\in C}$. As κ is regular, there exists $\alpha < \kappa$ such that $g(c), f(c) \in M_\alpha$ for all $c \in C$. This implies $(\mathcal{M}_\alpha, f(c))_{c\in C} \equiv (\mathcal{M}_\alpha, g(c))_{c\in C}$, that is, there exists an elementary map $\widetilde{h} : \{f(c : c \in C)\} \to \{g(c) : c \in C\}$ (as subsets of $\mathcal{M}_\alpha$) such that $\widetilde{h}(f(c)) = g(c)$ for all $c \in C$. Hence, the result above yields an elementary embedding $h_\alpha : \mathcal{M}_\alpha \to \mathcal{M}_{\alpha+1}$ such that the image of $f(c)$ is $g(c)$ for all $c \in C$. Now, we have that $(\mathcal{M}_{\alpha+1}, a)_{a\in M_\alpha} \equiv (\mathcal{M}_{\alpha+1}, h_\alpha(a))_{a\in M_\alpha}$ so the previous result yields an elementary embedding $h_{\alpha+1} : \mathcal{M}_{\alpha+1} \to \mathcal{M}_{\alpha+2}$ such that $h_{\alpha+1}(h_\alpha(a)) = a$ for all $a \in \mathcal{M}_\alpha$. We can keep this doing this construction in an analogous way, getting $h_{\alpha+2} : \mathcal{M}_{\alpha+2} \to \mathcal{M}_{\alpha+3}$ such that $h_{\alpha+2}(h_{\alpha+1}(a)) = a$ for all $a \in \mathcal{M}_{\alpha+1}$ and so on. Note that $h^{-1}_{\alpha+1}$ extends h_α, $h_{\alpha+2}$ extends $h^{-1}_{\alpha+1}$ and so on. In limit ordinal we take unions of pairs $h_\beta \cup h^{-1}_{\beta+1}$. Using this argument we proceed by induction to construct all h_β with $\alpha \leqslant \beta < \kappa$. Finally, taking the union of all the constructed pairs $h_\beta \cup h^{-1}_{\beta+1}$

we get the required isomorphism since each $h_\beta \cup h_{\beta+1}^{-1}$ is an elementary map with $\mathcal{M}_\beta$ contained in the domain and $\mathcal{M}_{\beta+1}$ contained in the image.

Let L' be a sublanguage of L. For each $\alpha < \kappa$ the reduct of $\mathcal{M}_{\alpha+1}$ to L' is τ_α saturated. Hence, we can apply an analogous argument to the given above to the reduct of $\mathcal{N}$ to L'. □

The next corollary follows immediately from the proposition above.

Corollary 1.2.24. *Every complete theory has a κ-saturated, strongly κ-homogeneous model for every infinite cardinal κ.*

1.3. The space of types

The space of types is a relevant part of both theories, the classical and the continuous one. We show in this section that in the continuous setting, when T is complete, this space is not just a topological space as in classical model theory, it is also a metric one.

We fix a signature L for metric structures and a complete L-theory T.

For $\mathcal{N}$, a model of T, and $A \subseteq N$. We denote the $L(A)$-structure $(\mathcal{N}, a)_{a\in A}$ by $\mathcal{N}_A$ and $T_A = Th(\mathcal{N}_A)$.

Let $x_1, \ldots, x_n$ be distinct variables. A set p of $L(A)$-conditions with all free variables among $x_1, \ldots, x_n$ is called an *n-type* over A if there exist a model $\mathcal{M}_A$ of T_A and $e_1, \ldots, e_n \in M$ such that p is the set of all $L(A)$-conditions $E(x_1 \ldots, x_n)$ for which $\mathcal{M}_A \models E[e_1, \ldots, e_n]$. In this case, we denote $p = \mathrm{tp}_{\mathcal{M}}(e_1, \ldots, e_n/A)$ and say that $(e_1, \ldots, e_n)$ *realizes* p in $\mathcal{M}$.

The collection of all such n-types over A is denoted $S_n(T_A)$, or simply $S_n(A)$ if the context makes T_A clear.

The following properties follows from the definition of type.

Remark 1.3.1. *Let $\mathcal{M}$ be an L-structure and, A be as subset of M. Let e, e' be n-tuples from M.*

1. $\mathrm{tp}_{\mathcal{M}}(e/A) = \mathrm{tp}_{\mathcal{M}}(e'/A)$ *if and only if* $(\mathcal{M}_A, e) \equiv (\mathcal{M}_A, e')$.
2. *If* $\mathcal{M} \preccurlyeq \mathcal{N}$, *then* $\mathrm{tp}_{\mathcal{M}}(e/A) = \mathrm{tp}_{\mathcal{N}}(e/A)$.

Remark 1.3.2. *Suppose $\mathcal{M}$ is a κ-saturated L-structure. Then, for any $A \subseteq M$ of cardinality strictly less than κ, every type in $S_n(T_A)$ is realized in $\mathcal{M}$ for every $n \geqslant 1$. In fact, this property is equivalent to κ-saturation of $\mathcal{M}$.*

Proof. Let $p \in S_n(T_A)$, by definition of type, there exist $\mathcal{N} \models T_A$ and $e_1, \ldots, e_n \in N$ such that $\mathcal{N} \models p[e_1, \ldots, e_n]$. Let $\Gamma(x_1 \ldots, x_n) = \{\varphi_1 = 0, \ldots, \varphi_n = 0\}$ be a finite subset of p. Then, $\mathcal{N} \models \Gamma[e_1, \ldots, e_n]$. This implies

$$\mathcal{N} \models \inf_{x_1} \ldots \inf_{x_n} (\bigwedge_{j=1}^{m} \varphi_j((x_1, \ldots, x_n))) = 0.$$

As this is an $L(A)$-sentence,

$$(\mathcal{M}, a)_{a\in A} \models \inf_{x_1} \dots \inf_{x_n} (\bigwedge_{j=1}^{m} \varphi_j((x_1, \dots, x_n))) = 0.$$

Finally, proposition 1.2.18 yields that there exist elements $e_1 \dots, e_n \in M$ such that $(\mathcal{M}, a)_{a\in A} \models \Gamma[e_1, \dots, e_n]$ and, as Γ was arbitrary, the κ-saturation of $\mathcal{M}$ ensures that p is realized in $\mathcal{M}$. The converse can be done as in the classical case. □

Now, we study the topology of the space of types.

The *logic topology* on $S_n(T_A)$ is defined as follows. If $p \in S_n(T_A)$, the basic open neighborhoods of p are the sets of the form

$$[\varphi < \varepsilon] = \{q \in S_n(T_A) : \varphi \leqslant \delta \text{ is in } q \text{ for some } 0 \leqslant \delta < \varepsilon\}$$

for which the condition $\varphi = 0$ is in p and $\varepsilon > 0$.

Note that sets of the form

$$[\varphi \leqslant \varepsilon] = \{q \in S_n(T_A) : \varphi \leqslant \varepsilon \text{ is in } q\},$$

where $\varphi(x_1, \dots, x_n)$ is an $L(A)$-formula and $\varepsilon \geqslant 0$, are closed. This follows from the fact that its compliment is $\varnothing$ if $\varepsilon \geqslant 1$ and $[1 \dot{-} \varphi < 1 - \varepsilon]$ otherwise.

Remark 1.3.3. *The logic topology on $S_n(T_A)$ is Hausdorff.*

Proof. If p and q are distinct types of $S_n(T_A)$, there exist an $L(A)$-formula φ such that $\varphi = 0$ is in p but not in q. Therefore, $\varphi = r$ is in q for some positive r. Taking $\varepsilon = \frac{r}{2}$, the sets $[\varphi < \varepsilon]$ and $[(r \dot{-} \varphi) < \varepsilon]$ are disjoint open sets one containing p and the other containing q. □

Lemma 1.3.4. *The closed subsets of $S_n(T_A)$ for the logic topology are exactly the sets of the form $C_\Gamma = \{p \in S_n(T_A) : \Gamma(x_1, \dots, x_n) \subseteq p\}$ where $\Gamma(x_1, \dots, x_n)$ is a set of $L(A)$-conditions.*

Proof. Given a set $\Gamma(x_1, \dots, x_n)$ of $L(A)$-conditions, note that C_Γ is the intersection of all sets $[\varphi \leqslant 0]$ where $\varphi = 0$ is any condition in Γ. Hence C_Γ is closed. Conversely, suppose $C \subset S_n(T_A)$ is closed in the logic topology and let $p \in S_n(T_A) \backslash C$. By the definition of the logic topology there exists an $L(A)$-condition $\varphi = 0$ in p and $\varepsilon > 0$ such that $[\varphi < \varepsilon]$ is disjoint from C. We may assume the nontrivial case $\varepsilon \leqslant 1$. Then the closed $[(\varepsilon \dot{-} \varphi) \leqslant 0]$ contains C and does not have p as an element. We can represent C as the set of all types containing all the conditions of the form $(\varepsilon \dot{-} \varphi) = 0$ with $\varphi = 0 \in p$ and $\epsilon > 0$. □

Proposition 1.3.5. *For any $n \geqslant 1$, $S_n(T_A)$ is compact with respect to the logic topology.*

Proof. By the previous result, we already know that closed sets can be expressed as $C_\Gamma = \{p \in S_n(T_A) : \Gamma(x_1, \ldots, x_n) \subseteq p\}$, where $\Gamma(x_1, \ldots, x_n)$ is a set of $L(A)$-conditions. Hence, $C_{\Gamma_1} \cap C_{\Gamma_2} = C_{\Gamma_1 \cup \Gamma_2}$. Suppose that the family $\{C_{\Gamma_i} : i \in I\}$ has the finite intersection property, this implies that every finite subset of $\bigcup\{\Gamma_i : i \in I\}$ is consistent with T_A. Then, compactness theorem 1.2.11 yields that the entire set is consistent with T_A. Therefore, there exists at least one $p \in S_n(T_A)$ such that $\bigcup\{\Gamma_i : i \in I\} \subseteq p$. Therefore, $\bigcap_{i \in I} C_{\Gamma_i} \neq \varnothing$. □

Now, we define a metric on the space of types $S_n(T_A)$. For each $n \geqslant 1$ the metric is defined as a quotient of the given metric d in M^n, where M is the universe of a suitable model $\mathcal{M}_A \models T_A$. We also denote this metric by d.

This suitable model is any $\mathcal{M}_A \models T_A$ where $\mathcal{M}$ is a κ-saturated model of T with $\kappa > |A|$. Therefore, each type in $S_n(T_A)$ is realized for each $n \geqslant 1$. Let (M, d) be the underlying metric space of $\mathcal{M}$. For $p, q \in S_n(T_A)$, we define $d(p, q)$ to be

$$\inf\{\max_{i \leqslant j \leqslant n} d(b_j, c_j) : \mathcal{M}_A \models p[b_1, \ldots b_n], \mathcal{M}_A \models q[c_1, \ldots, c_n]\}.$$

Note that this expression does not depend on $\mathcal{M}_A$, since $\mathcal{M}_A$ realizes every type of a $2n$-tuple $(b_1, \ldots, b_n, c_1, \ldots, c_n)$ over A.

Lemma 1.3.6. *The infimum above is always attained at a pair of points $a, b \in M^n$ such that $\mathcal{M}_A \models p[a]$, $\mathcal{M}_A \models q[b]$.*

Proof. Since the set of conditions $p(x) \cup q(y) \cup \{d(x, y) \leqslant d(p, q) + \frac{1}{n} : n \geqslant 1\}$ is finitely satisfiable, the saturation of $\mathcal{M}$ implies that there exists an element satisfying the whole set. Hence, if $d(p, q) = 0$, they have a common realization, so $p = q$. □

Theorem 1.3.7. *The distance d defined above, defines a metric in $S_n(T_A)$.*

Proof. The only property of a metric that is nontrivial is the triangular inequality. To prove it, we first show that given $a' \in M^n$ such that $\mathcal{M}_A \models p[a']$, there exists $b' \in M^n$, $\mathcal{M}_A \models q[b']$, such that $d(a', b') = d(p, q)$. Suppose that $d(p, q)$ is attained in a pair of points $a, b \in M^n$. As a and a' both realize the same type p over A, there exists an elementary map

$$h : A \cup \{a\} \to A \cup \{a'\}.$$

Now, we use that κ-saturation implies κ-homogeneity 1.2.21 to extend this elementary function to an elementary map

$$h' : A \cup \{a, b\} \to A \cup \{a', b'\} \subseteq M.$$

Hence $\mathcal{M} \models q[b']$ and $d(a, b) = d(a', b') = d(p, q)$.

Now, let $p, q, r \in S_n(T_A)$. Suppose that $d(p, r) = d(a, c_1)$ and $d(r, q) = d(c_2, b)$. As, c_1 and c_2 realize the same type r over A, by the discussion above, there exist b', $\mathcal{M} \models q[b']$, such that $d(c_1, b') = d(c_2, b)$. Hence,

$$d(p, r) + d(r, q) = d(a, c_1) + d(c_1, b') \geqslant d(a, b') \geqslant d(p, q),$$

Where the middle inequality is just the tiangular inequality for the metric on M^n. □

Proposition 1.3.8. *The d-topology is finer than the logic topology on $S_n(T_A)$*

Proof. It suffices to prove that there is a d-open set inside each basic open set of the logic topology. Let $\mathcal{M}_A$ be as above and let $[\varphi < \varepsilon]$ with $\varepsilon > 0$ be a basic open neighborhood of $p \in S_n(T_A)$. By the uniform continuity of formulas, there exist $\delta > 0$ such that if $x, y \in M^n$ satisfy $d(x, y) < \delta$, then $|\varphi^{\mathcal{M}}(x) - \varphi^{\mathcal{M}}(y)| < \varepsilon$. It follows easily that the open ball around p of radius δ with respect to the d-metric is contained in $[\varphi < \varepsilon]$. □

Proposition 1.3.9. *The metric space $(S_n(T_A), d)$ is complete.*

Proof. Let $(p_k)_{k \geqslant 1}$ be a Cauchy sequence in $(S_n(T_A), d)$. Without loss of generality we may assume $d(p_k, p_{k+1}) \leqslant 2^{-k}$ for all k. Let $\mathcal{N}$ be an ω-saturated and strongly ω-homogeneous model of T_A.

Without loss of generality we may assume $\mathcal{N} = \mathcal{M}_A$ for some $\mathcal{M} \models T$. We claim that for any $a' \in \mathcal{N}$ such that $\mathcal{N} \models p_k[a']$, there exists $b' \in \mathcal{N}$ satisfying $\mathcal{N} \models p_{k+1}[b']$ such that $d(a', b') = d(p_k, p_{k+1})$. Indeed, let a and b the elements of $\mathcal{N}$ such that $d(a, b) = d(p_k, p_{k+1})$, the existence of this elements is guaranteed by 1.3.6 . Since a and a' satisfy the same type over A, we have that there exists an elementary map

$$h : \{a\} \to \{a'\} \subseteq N$$

As $\mathcal{N}$ is strongly ω-homogeneous, we can extend this elementary map to an isomorphism $f : (\mathcal{N}, a) \to (\mathcal{N}, a')$. Setting $b' = f(b)$ we have that b' satisfy the same type over A as b and that $d(a', b') = d(a, b) = d(p_k, p_{k+1})$. Therefore, proceeding inductively we may generate a sequence $(b_k)_{k \geqslant 1}$ in M^n such that $d(b_k, b_{k+1}) \leqslant 2^{-k}$ for all k. This implies that (b_k) is a Cauchy sequence in M^n so it has a limit $b \in M^n$. It follows that $\mathrm{tp}_{\mathcal{N}}(b)$ is the limit of (p_k) in $(S_n(T_A), d)$. □

We have proved that L-formulas defined functions from L-structures to $[0, 1]$. Now, we prove that formulas can be used to define functions from the space of types to $[0, 1]$.

Let $\mathcal{M}_A \models T_A$ be a model where every type in $S_n(T_A)$ is realized for all $n \geqslant 1$. For any $\varphi(x_1 \ldots, x_n)$ L-formula, we define a function $\widetilde{\varphi} : S_n(T_A) \to [0, 1]$ as $\widetilde{\varphi}(p) = \varphi^{\mathcal{M}}(b)$, where b is any realization of p in $\mathcal{M}_A$. This function is well defined because there is only one condition of the form $\varphi = r$ in each type.

Lemma 1.3.10. *Let $\varphi(x_1, \ldots, x_n)$ be any $L(A)$-formula. The function $\widetilde{\varphi} : S_n(T_A) \to [0, 1]$ is continuous for the logic topology and uniformly continuous for the d-metric in $S_n(T_A)$.*

Proof. Let $r \in [0, 1]$ and $\varepsilon > 0$, we check that

$$\widetilde{\varphi}^{-1}(r - \varepsilon, r + \varepsilon) = [|\varphi - r| < \varepsilon].$$

Remember that $[|\varphi - r| < \varepsilon] = \{p \in S_n(T_A) : |\varphi - r| \leqslant \delta \text{ is in } p \text{ for some } \delta < \varepsilon\}$. So if we take $t \in (r - \varepsilon, r + \varepsilon)$, $\widetilde{\varphi}^{-1}(t) = \{p \in S_n(T_A) : \varphi - t = 0 \text{ is in } p\} \subseteq [|\varphi - r| < \varepsilon]$.

On the other hand, if $p \in [|\varphi - r| < \varepsilon]$ then $\varphi = r + \delta$ is in p for some δ satisfying $|\delta| < \varepsilon$.

Hence $\widetilde{\varphi}$ is continuous for the logic topology. To prove the uniform continuity, remember that by remark 1.1.2, there exists a modulus of uniform continuity Δ_φ for $\varphi^{\mathcal{M}}$. We claim that Δ_φ is also a modulus of uniform continuity for $\widetilde{\varphi}$. Indeed, take $\varepsilon \in (0,1]$ and let $\delta = \Delta_\varphi(\varepsilon)$. Suppose $p, q \in S_n(T_A)$ satisfy $d(p,q) < \delta$. Take $a, b \in M^n$ such that $\mathcal{M} \models p[a]$, $\mathcal{M} \models q[b]$ and $d(a,b) = d(p,q)$. Then, by our choice of Δ_φ, we have

$$|\widetilde{\varphi}(p) - \widetilde{\varphi}(q)| = |\varphi^{\mathcal{M}}(a) - \varphi^{\mathcal{M}}(b)| \leqslant \varepsilon.$$

□

Proposition 1.3.11. *For any function $\Phi : S_n(T_A) \to [0,1]$ the following are equivalent:*

(1) *Φ is continuous for the logic topology on $S_n(T_A)$.*

(2) *There exist a sequence $(\varphi_k(x_1, \ldots, x_n))_{k \geqslant 1}$ of $L(A)$-formulas such that the sequence $(\widetilde{\varphi}_k)_{k \geqslant 1}$ converges uniformly to Φ on $S_n(T_A)$.*

(3) *Φ is continuous for the logic topology and uniformly continuous for the d-metric on $S_n(T_A)$.*

Proof. (3) $\Longrightarrow$ (1) is trivial.

(1) $\Longrightarrow$ (2): We are going to apply the following lattice version of the Stone-Weierstrass theorem:

'Let X be compact, and let A be a sublattice of $C(X)$, the real valued continuous functions on X. Then $cl(A)$, with the supremum distance, contains every function f in $C(X)$ that can be approximated at each pair of points in X by a function from A.'

It is easy to see that the functions of the form $\widetilde{\varphi}$ form a sublattice of $C(X)$ that contains constant functions $r \in [0,1]$ and separates points, i.e. for every $p, q \in S_n(T_A)$ with $p \neq q$, there exists $\widetilde{\varphi}$ such that $\widetilde{\varphi}(p) \neq \widetilde{\varphi}(q)$. To apply the result, we need to approximate an arbitrary continuous function $\Phi : S_n(T_A) \to [0,1]$ in an arbitrary pair of points $x, y \in S_n(T_A)$.

If $x = y$ we can take the constant function $r = \Phi(x)$.

If $x \neq y$, as the lattice separates points, there exists $\widetilde{\psi}$ such that $\widetilde{\psi}(x) = a'$ and $\widetilde{\psi}(y) = b'$ with $a' \neq b'$. Let $\Phi(x) = a$ and $\Phi(y) = b$. We have several cases

$a' < b'$. $f = \frac{\widetilde{\psi} \dot- a'}{b' \dot- a'}$ satisfies $f(x) = 0$ and $f(y) = 1$. So if $a \leqslant b$, then $(b \dot- a)f + a$ approximates Φ at x and y, and if $b \leqslant a$, then $(a \dot- b)(1 \dot- f) + b$ approximates Φ at x and y.

$b' < a'$. We use $f = \frac{\widetilde{\psi} \dot- b'}{a' \dot- b'}$ in an analogous manner.

Where we have considered that the operations $a + b$ and $\frac{a}{b}$ have their range restricted to $[0,1]$. This implies that functions of the form $\widetilde{\varphi}$ are dense in the set of $[0,1]$-valued continuous functions on $S_n(T_A)$ which implies (2).

(2) $\Longrightarrow$ (3). The properties of uniform continuity for the d-metric and continuity for the logic topology are both preserved by uniform continuity. Hence, as each $\widetilde{\varphi}_k$ has these properties, so does their uniform limit ☐

Proposition 1.3.12. *Let $\mathcal{M} \models T$ and $A \subseteq B \subseteq M$ and let $\pi : S_n(T_B) \to S_n(T_A)$ be the restriction map. Then*

(1) *π is surjective.*

(2) *π is continuous for the logic topologies.*

(3) *π is uniformly continuous for the d-metrics.*

(4) *If A is dense in B, then π is a homeomorphism for the logic topologies and a surjective isometry for the d-metrics.*

Proof. (1). Let $p \in S_n(T_A)$. The set of $L(A)$-formulas p^+ is finitely satisfiable in $(\mathcal{M}, b)_{b\in B}$. Hence, using compactness theorem 1.2.11, p is realized in some elementary extension of $(\mathcal{M}, b)_{b\in B}$. If $(e_1, \ldots, e_n)$ realize p in such elementary extension, then $p = \pi(\mathrm{tp}(e_1, \ldots, e_n/B))$.

(2). If $\varphi(x_1, \ldots, x_n)$ is an $L(A)$-formula and $\varepsilon > 0$, it is clear that the preimage of $[\varphi < \varepsilon]$ as a basic neighborhood in $S_n(T_A)$ is $[\varphi < \varepsilon]$ as a basic neighborhood in $S_n(T_B)$. Hence π is continuous.

(3). Any realization of $p \in S_n(T_B)$ is a realization of $\pi(p)$. Hence, $d(\pi(p), \pi(q)) \leqslant d(p, q)$, that means that π is a non-expansive function for the d-metrics, which implies that is uniformly continuous.

(4) As A is dense in B, the uniform continuity of formulas implies that any $L(B)$-formula can be approximated by a sequence of $L(A)$-formulas. Hence, if $\varphi = r \in p$, there exist a sequence $\varphi_k = r_k$ of $L(A)$-conditions in p with r_k converging to r. This implies that any realization of $\pi(p)$ is also a realization of p. As π is continuous for the logic topology, is also closed since $S_n(T_B)$ is compact and $S_n(T_A)$ is Hausdorff. Since we proved that π is inyective, then we have the properties required. ☐

Remark 1.3.13. *When T is not a complete theory, we consider types over the empty set. In this case, all the results are exactly the same, excluding the ones about the d-metric on types. The distance between types can be modified to allow T not to be complete by setting $d(p, q) = \infty$ if p and q belong to different completions of T.*

CHAPTER 2

Definability, algebraic and definable closures

In this chapter we develop all the concepts related with definability in continuous model theory. We start with predicates and construct the rest of the notions from that one.

2.1. Definability

2.1.1. Predicates

We say that an n-ary predicate P is *definable* in $\mathcal{M}$ over A if there exist a sequence of $L(A)$-formulas $(\varphi_k : k \geqslant 1)$ such that $(\varphi_k^{\mathcal{M}}(x) : k \geqslant 1)$ converges uniformly to P on M^n. It is clear that definable predicates are uniformly continuous since they are the uniform limit of a sequence of uniformly continuous functions.

Lemma 2.1.1. *If P and Q are definable predicates in $\mathcal{M}$ over A, then $\inf_x P(x)$ is also a definable predicate in $\mathcal{M}$ over A and so is $u(P, Q)$ for each connective u.*

Proof. Let $(\varphi_k : k \geqslant 1)$ be the sequence of formulas such that $(\varphi_k^{\mathcal{M}} : k \geqslant 1)$ converges uniformly to P on M^n and let $(\psi_k : k \geqslant 1)$ be the sequence of formulas such that $(\varphi_k^{\mathcal{M}} : k \geqslant 1)$ converges uniformly to Q on M^m. Properties of uniform convergence of uniformly continuous functions ensure that the sequence of uniformly continuous function $(\inf_x \varphi_k^{\mathcal{M}} : k \geqslant 1)$ converges uniformly to $\inf_x P(x)$ on M^n and that the sequence $(u(\varphi_k, \psi_k)^{\mathcal{M}} : k \geqslant 1)$ converges uniformly to $u(P, Q)$ on M^{n+m}. □

Instead of working with uniform limits, one could broad the definition of connective. In order to do that, we consider the space $[0,1]^{\mathbb{N}}$ equipped with the distance $\widetilde{d}((a_k),(b_k)) = \sum_{k=0}^{\infty} 2^{-k}|a_k - b_k|$. If we allow continuous functions $u : [0,1]^{\mathbb{N}} \to [0,1]$ to be connectives, we can reformulate our definition without causing any problem as the following result shows.

Proposition 2.1.2. *Let $\mathcal{M}$ be an L-structure and let $A \subseteq M$. Suppose P is an n-ary predicate. Then, P is definable in $\mathcal{M}$ over A if and only if there exist a continuous function $u : [0,1]^{\mathbb{N}} \to [0,1]$ and a sequence of $L(A)$-formulas $(\varphi_i : i \in \mathbb{N})$ such that*

$$P(x) = u(\varphi_i^{\mathcal{M}}(x) : i \in \mathbb{N})$$

for all $x \in M^n$.

Proof. Suppose $P(x)$ has the form $u(\varphi_i^{\mathcal{M}}(x) : i \in \mathbb{N})$. We need to show that for every $\varepsilon > 0$ there exists an $L(A)$-formula φ_ε such that $|P^{\mathcal{M}}(a) - \varphi_\varepsilon^{\mathcal{M}}(a)| \leqslant \varepsilon$ for all $a \in M^n$. Fixing $\varepsilon > 0$, we are going to show that there exists an $L(A)$-formula φ such that $|P^{\mathcal{M}}(a) - \varphi^{\mathcal{M}}(a)| \leqslant \varepsilon$ for all $a \in M^n$.

The compacity of $([0,1]^{\mathbb{N}}, \rho)$ implies that u is uniformly continuous. Hence if the sequences $(a_k : k \in \mathbb{N})$ and $(b_k : k \in \mathbb{N})$ are such that $a_k = b_k$ for all $k = 0, \dots, m$ for a large enough m we have that

$$|u((a_k)_{k \in \mathbb{N}}) - u((b_k)_{k \in \mathbb{N}})| \leqslant \varepsilon.$$

For that large enough m, let us define $u_m : [0,1]^{m+1} \to [0,1]$ as

$$u_m(a_0, \dots, a_m) = u(a_0, \dots, a_m, 0, 0, \dots)$$

for all $a_0, \dots, a_m \in [0,1]$. The continuity of u implies the continuity of u_m, hence u_m is a connective. Let $\varphi(x)$ be the $L(A)$-formula defined by

$$\varphi(x) = u_m(\varphi_0(x), \dots, \varphi_m(x)).$$

Then, it is clear that we have

$$|P^{\mathcal{M}}(a) - \varphi^{\mathcal{M}}(a)| = |u((\varphi_k(a))_{k \in \mathbb{N}}) - u_m(\varphi_0(a), \dots, \varphi_m(a))| \leqslant \varepsilon$$

for all $a \in M^n$. Hence P is definable in $\mathcal{M}$ over A.

For the converse, we note that $([0,1]^{\mathbb{N}}, \rho)$ is a normal topological space and hence we have the Tietze extension theorem:

'If X is a normal topological space and $A \subseteq X$ a closed subset. For every continuous function $f : A \to \mathbb{R}$, there exists a continuous function $F : X \to \mathbb{R}$ extending f.'

Assuming that $P : M^n \to [0,1]$ is definable in $\mathcal{M}$ over A, for every $k \in \mathbb{N}$, let φ_k be an $L(A)$-formula such that

$$|\varphi_k^{\mathcal{M}}(x) - P(x)| \leqslant 2^{-k}$$

for all $x \in M^n$.

Consider the set C of all sequences $(a_k : k \in \mathbb{N})$ in $[0,1]$ satisfying $|a_k - a_l| \leqslant 2^{-N}$ whenever $N \in \mathbb{N}$ and $k, l \geqslant N+1$. We claim that set C is a closed subset of $[0,1]^{\mathbb{N}}$ and also a subset of the Cauchy sequences in $[0,1]$, the latter implies that every sequence in C has a limit. To prove the claim, note that if $(a_k : k \geqslant 1) \notin C$ then there exists some $N \in \mathbb{N}$ and some $k, l \geqslant N+1$ such that $|a_k - a_l| > 2^{-N}$, taking $\delta > 0$ small enough, it is easy to see that every $(c_k : k \geqslant 1) \in B_\delta((a_k))$ also satisfies $|c_k - c_l| > 2^{-N}$. It is also easy to check that $(\varphi_k^{\mathcal{M}}(x) : k \in \mathbb{N}) \in C$ for every $x \in M^n$. We also claim that the function $\lim : C \to [0,1]$ is a continuous function with respect to the subspace topology of C. To prove this claim one can use that in sequences like the ones we are considering, the distance between the n coordinate and the limit is bounded, also noting that for δ small enough, if $\widetilde{d}((a_k),(c_k)) < \delta$ then c_n and a_n are close. Finally, Tietze Extension Theorem yields a continuous function $u : [0,1]^{\mathbb{N}} \to [0,1]$ that agrees with lim on C. The latter u is the required connective. □

We study now the relation between definable predicates and elementary extensions or substructures.

Proposition 2.1.3. *Let $P_1, \dots, P_m$ be n-ary definable predicates in $\mathcal{M}$ over A. Let $\mathcal{N} \preccurlyeq \mathcal{M}$ with $A \subseteq N$ and let Q_i be the restriction of P_i to $\mathcal{N}$ for each i. Then, the following holds:*

$$(\mathcal{N}, Q_1, \dots, Q_m) \preccurlyeq (\mathcal{M}, P_1, \dots, P_m).$$

Proof. We first note the following two properties of predicates related to uniform convergence of formulas:

Firstly, if P is a definable predicate in $\mathcal{M}$ over A and Q is its restriction to $\mathcal{N}$ we claim that $\inf_x P(x) = \inf_x Q(x)$, where $\inf_x$ is taken over M^n in the left hand side and over N^n on the right hand side. This is because if $(\varphi_k : k \geqslant 1)$ is the set of formulas whose interpretations converge to P in M^n we have

$$\inf_x P(x) = \lim_{k\to\infty} \inf_x \varphi_k^{\mathcal{M}}(x) = \lim_{k\to\infty} \inf_x \varphi_k^{\mathcal{N}}(x) = \inf_x Q(x),$$

where the second equality holds because properties of uniform limits allow us to commute the limit and the infimum.

Secondly, if $P_1, \dots, P_m$ are definable predicates in $\mathcal{M}$ over A. Let $(\varphi_k^i : k \geqslant 1)$ the sequence of formulas whose interpretations converge to P_i on M^n and let u be an m-ary connective. Using properties of uniform convergence we have

$$u(P_1(x_1), \dots, P_m(x_m)) = \lim_{k\to\infty} u(\varphi_k^{1^{\mathcal{M}}}(x_1), \dots, \varphi_k^{m^{\mathcal{M}}}(x_m)).$$

From this two observations and from the fact that predicates can only be in the scope of a quantifier or in composition with a connective, it follows that given a formula $\varphi^{\mathcal{M}}(P_1(x_1), \dots, P_m(x_m), x)$ we have that whenever $a_1, \dots, a_m, a$ are tuples of elements in N

$$\varphi^{\mathcal{M}}(P_1(a_1), \dots, P_m(a_m), a) = \lim_{k\to\infty} \varphi^{\mathcal{M}}(\varphi_k^{1^{\mathcal{M}}}(a_1), \dots, \varphi_k^{m^{\mathcal{M}}}(a_m), a) =$$
$$= \lim_{k\to\infty} \varphi^{\mathcal{N}}(\varphi_k^{1^{\mathcal{N}}}(a_1), \dots, \varphi_k^{m^{\mathcal{N}}}(a_m), a) = \varphi^{\mathcal{N}}(Q_1(a_1), \dots, Q_m(a_m), a).$$

□

Definable predicates are well behaved when passing to elementary extensions or substructures as the next results show.

Proposition 2.1.4. *Let P be an n-ary predicate definable in $\mathcal{M}$ over A and let $\mathcal{N}$ be such that $\mathcal{M} \preccurlyeq \mathcal{N}$. Then there exists an unique n-ary definable predicate in $\mathcal{N}$ over A, Q, that extends P. Furthermore, $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$.*

Proof. Before proving the existence, let us show that if such predicate Q exists, it must be unique.

Suppose Q_1, Q_2 are predicates definable in $\mathcal{N}$ over A whose restrictions to M^n is equal to P. Applying proposition 2.1.3, we get that $(\mathcal{M}, P, P) \preccurlyeq (\mathcal{N}, Q_1, Q_2)$ and hence

$$\sup\{|Q_1(x) - Q_2(x)| : x \in N^n\} = \sup\{|P(x) - P(x)| : x \in M^n\} = 0.$$

Therefore, $Q_1 = Q_2$.

To prove the existence, let $(\varphi_k : k \in \mathbb{N})$ a sequence of $L(A)$-formulas converging uniformly to P on M^n. Since $\mathcal{M} \preccurlyeq \mathcal{N}$, for any $k, l \in \mathbb{N}$ we have

$$\sup\{|\varphi_k^{\mathcal{N}}(b) - \varphi_l^{\mathcal{N}}(b)| : b \in N^n\} = \sup\{|\varphi_k^{\mathcal{M}}(a) - \varphi_l^{\mathcal{M}}(a)| : a \in M^n\}.$$

Hence $(\varphi_k^{\mathcal{N}} : k \in \mathbb{N})$ is a Cauchy sequence of functions that converges uniformly on N^n to some function $Q : N^n \to [0, 1]$. It is clear that Q extends P, and by construction is definable in $\mathcal{N}$ over A. The statement $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$ follows immediately applying proposition 2.1.3. □

The following result gives a characterization of definable predicates with respect to continuous functions on the type space.

Theorem 2.1.5. *Let $P : M^n \to [0, 1]$ be a function. Then, P is a definable predicate in $\mathcal{M}$ over A if and only if there exists a continuous function (with respect to the logic topology) $\Phi : S_n(Th(\mathcal{M}_A)) \to [0, 1]$ satisfying $P(a) = \Phi(\mathrm{tp}_{\mathcal{M}}(a/A))$ for all $a \in M^n$.*

Proof. If there exists such continuous function Φ. We know by proposition 1.3.11 that there exists a sequence $(\varphi_k : k \geqslant 1)$ of $L(A)$-formulas such that the sequence of functions $(\widetilde{\varphi}_k^{\mathcal{M}} : k \geqslant 1)$ (where each function is defined as in 1.3) converges uniformly to Φ on $S_n(Th(\mathcal{M}_A))$. Then, for any $a \in M^n$ let $p = \mathrm{tp}_{\mathcal{M}}(a/A)$. We have, by definition of the function $\widetilde{\varphi}_k$, the equality $\varphi_k^{\mathcal{M}}(a) = \widetilde{\varphi}_k(p)$ for all $k \geqslant 1$, and so

$$|\varphi_k^{\mathcal{M}}(a) - P(a)| = |\widetilde{\varphi}_k(p) - \Phi(p)|.$$

Hence, the functions $(\varphi_k^{\mathcal{M}} : k \geqslant 1)$ converge uniformly to P on M^n, that is, P is definable in $\mathcal{M}$ over A.

For the converse, suppose that $(\varphi_k : k \geqslant 1)$ is a sequence of $L(A)$-formulas such that the functions $(\varphi_k^{\mathcal{M}} : k \geqslant 1)$ converge uniformly to P on M^n. Let $\mathcal{N}$ be a κ-saturated elementary extension of $\mathcal{M}$ with $|A| < \kappa$. Since $\mathcal{M} \preccurlyeq \mathcal{N}$, for any $k, l \geqslant 1$ we have

$$\sup\{|\varphi_k^{\mathcal{N}}(b) - \varphi_l^{\mathcal{N}}(b)| : b \in N^n\} = \sup\{|\varphi_k^{\mathcal{M}}(a) - \varphi_l^{\mathcal{M}}(a)| : a \in M^n\}.$$

Hence $(\varphi_k^{\mathcal{N}} : k \in \mathbb{N})$ is a Cauchy sequence of functions that converges uniformly on N^n to some function $Q : N^n \to [0,1]$. The predicate Q extends P and is definable in $\mathcal{N}$ over A since is the uniform limit of the same sequence of formulas as P. Let p be any type in $S_n(T_A)$ and define $\Phi(p) = Q(b)$, where $b \in N^n$ is any realization of p. Since $Q(b)$ is the limit of a sequence of $L(A)$-formulas, the value of $Q(b)$ only depends on $\mathrm{tp}_{\mathcal{N}}(b/A)$. We claim that Φ is the uniform limit of $(\widetilde{\varphi}_k : k \geqslant 1)$ on $S_n(T_A)$. To prove this, we note that for any $p \in S_n(T_A)$ and all $k \geqslant 1$ we have

$$|\widetilde{\varphi}_k(p) - \Phi(p)| = |\varphi_k^{\mathcal{N}}(b) - Q(b)|,$$

where $b \in N^n$ is any realization of p, by the saturation hypothesis on $\mathcal{N}$ always exists at least one realization of p. Since $(\widetilde{\varphi}_k : k \geqslant 1)$ is a sequence of continuous functions, Φ is continuous with respect to the logic topology (see 1.3.11). Finally for any $a \in M^n$ we have $P(a) = Q(a) = \Phi(\mathrm{tp}_{\mathcal{N}}(a/A)) = \Phi(\mathrm{tp}_{\mathcal{M}}(a/A))$ as required. □

We now give a characterization of definable predicates for saturated models.

Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. A set $S \subseteq M^n$ is called *type-definable* in $\mathcal{M}$ over A if there exists a set of $L(A)$-formulas $\Sigma(x_1, \ldots, x_n)$ (or a set of $L(A)$-conditions $E = \{\varphi = 0 : \varphi \in \Sigma)\})$ such that for any $a \in M^n$ we have $a \in S$ if and only if $\varphi^{\mathcal{M}}(a) = 0$ for every $\varphi \in \Sigma$.

Corollary 2.1.6. *Let $\mathcal{M}$ be a κ-saturated structure and $A \subseteq M$ with $|A| < \kappa$. Let $P : M^n \to [0,1]$ be a function. Then P is a predicate definable in $\mathcal{M}$ over A if and only if the sets $\{a \in M^n : P(a) \leqslant r\}$ and $\{a \in M^n : P(a) \geqslant r\}$ are type-definable in $\mathcal{M}$ over A for every $r \in [0,1]$.*

Proof. Suppose P is a definable predicate in $\mathcal{M}$ over A. Theorem 2.1.5 gives us a continuous function $\Phi : S_n(T_A) \to [0,1]$ such that $P(a) = \Phi(\mathrm{tp}_{\mathcal{M}}(a/A))$ for all $a \in M^n$. Fix $r \in [0,1]$. The sets $\Phi^{-1}([0,r])$ and $\Phi^{-1}([r,1])$ are closed subsets of $S_n(T_A)$ for the logic topology. Due to lemma 1.3.4, we know that those sets have the form

$$\Phi^{-1}([0,r]) = \{p \in S_n(T_A) : \Gamma_1(x_1, \ldots, x_n) \subseteq p\}$$
$$\text{and}$$
$$\Phi^{-1}([r,1]) = \{p \in S_n(T_A) : \Gamma_2(x_1, \ldots, x_n) \subseteq p\},$$

where Γ_1 and Γ_2 are sets of $L(A)$-conditions. It follows that $\{a \in M^n : P(a) \leqslant r\}$ is type-defined in $\mathcal{M}$ over A by the set of formulas Γ_1 and $\{a \in M^n : P(a) \geqslant r\}$ is type-defined in $\mathcal{M}$ over A by the set of formulas Γ_2

For the converse, let P be a function such that the sets $\{a \in M^n : P(a) \leqslant r\}$ and $\{a \in M^n : P(a) \geqslant r\}$ are type-definable in $\mathcal{M}$ over A for every $r \in [0,1]$. This allows us to define $\Phi : S_n(T_A) \to [0,1]$ by setting $\Phi(p) = P(a)$ whenever $p \in S_n(T_A)$and $a \in M^n$ realizes p. The existence of this element a is guaranteed by the saturation of $\mathcal{M}$. We claim that Φ is well defined and continuous for the logic topology. Indeed, to prove continuity, we only need to check that the preimages of closed intervals are closed sets. We note first that $\Phi^{-1}([r_1, r_2]) = \Phi^{-1}([0, r_2]) \cap \Phi^{-1}([r_1, 1])$ and that the sets $\{a \in M^n : P(a) \leqslant r_2\}$ and $\{a \in M^n : P(a) \geqslant r_1\}$ are type-definable in $\mathcal{M}$ over A by some sets of $L(A)$-conditions Γ_1 and Γ_2. Now let $p \in S_n(T_A)$. We have $p \in \Phi^{-1}([0, r_2])$ if and only if any realization $a \in M^n$ of p satisfies $P(a) \leqslant r_2$, that is, if and only if $\mathcal{M} \models \Gamma_2[a]$, it follows that $\Phi^{-1}([0, r_2]) = \{p \in S_n(T_A) : \Gamma \subseteq p\}$ which is closed by lemma 1.3.4. An analogous argument applies to $\Phi^{-1}([r_1, 1])$. From what we have proved it also follows that Φ is well defined since we can express $\Phi^{-1}(\{r\})$ as $\Phi^{-1}([0, r]) \cap \Phi^{-1}([r, 0])$. After Φ is constructed, we can apply theorem 2.1.5, this ensures that P is definable in $\mathcal{M}$ over A. □

The following corollary is the analogous to the Theorem of Svenonius from classical model theory.

Corollary 2.1.7. *Let $\mathcal{M}$ be an L-structure with $A \subseteq M$ and let $P : M^n \to [0, 1]$ be a predicate. Then, P is definable in $\mathcal{M}$ over A if and only if whenever for any elementary extension $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$, the predicate Q is invariant under all automorphisms of $\mathcal{N}$ that leaves A fixed pointwise.*

Proof. First, assume that P is definable in $\mathcal{M}$ over A. Let $(\varphi_k : k \geqslant 1)$ be a sequence of $L(A)$-formulas such that $(\varphi_k^{\mathcal{M}} : k \geqslant 1)$ converges uniformly to P in M^n. If $(\mathcal{N}, Q)$ is an elementary extension of $(\mathcal{M}, P)$, then the proof of proposition 2.1.4 implies that the predicate Q is the uniform limit of $(\varphi_k^{\mathcal{N}} : k \geqslant 1)$ on N^n. Since each $\varphi_k^{\mathcal{N}}$ is invariant under all automorphisms of $\mathcal{N}$ that fix A pointwise, the uniform limit of $(\varphi_k^{\mathcal{N}} : k \geqslant 1)$ also is invariant under all automorphisms of $\mathcal{N}$ that fix A pointwise.

For the converse, let $(\mathcal{N}, Q)$ be an elementary extension of $(\mathcal{M}, P)$ such that $\mathcal{N}$ is strongly κ-homogeneous and $(\mathcal{N}, Q)$ is κ-saturated with $|A| < \kappa$. We define $\Phi : S_n(T_A) \to [0, 1]$ by $\Phi(p) = Q(b)$, where b is any element of N^n realizing p. We are going to check that this function is well-defined and continuous in order to apply theorem 2.1.5. Note first that our saturation hypothesis implies that every $p \in S_n(T_A)$ is realized in $\mathcal{N}_A$ and that the homogeneity hypothesis implies that $Aut_A(\mathcal{N})$ acts transitively on the set of realizations of any $p \in S_n(T_A)$. Hence, since Q is $Aut_A(\mathcal{N})$-invariant, the function Φ is well defined. Now we are going to check that Φ is continuous for the logic topology on $S_n(T_A)$. Fix $p \in S_n(T_A)$ and let $r = \Phi(p)$. Since all realizations b of p in N^n satisfy $Q(b) = r$, the κ-saturation of $(\mathcal{N}, Q)$ implies that for every $\varepsilon > 0$ there exists a condition $\varphi = 0$ in p and $\delta > 0$ such that for any $b \in N^n$ $\varphi^{\mathcal{N}}(b) < \delta$ implies $|Q(b) - r| \leqslant \frac{\varepsilon}{2}$. Therefore for every $p \in \Phi^{-1}(r - \varepsilon, r + \varepsilon)$ there exists an open neighbourhood with respect to the logic topology $[\varphi < \delta]$ of p contained in $\Phi^{-1}(r - \varepsilon, r + \varepsilon)$. Hence Φ is continuous. Finally, by theorem 2.1.5 we conclude that P is definable in $\mathcal{M}$ over A. □

Now, we show that distance predicates can be axiomatized in continuous logic.

Theorem 2.1.8. *Let $(\mathcal{M}, P)$ be an L-structure satisfying*

$$\sup_x \inf_y \max(P(y), |P(x) - d(x, y)|) = 0$$

and

$$\sup_x |P(x) - \inf_y \min(P(y) + d(x, y), 1)| = 0$$

and let $D = \{x \in M^n : P(x) = 0\}$. *Then,* $P(x) = \text{dist}(x, D)$ *for all* $x \in M^n$

Proof. Due to the second condition, we have that $P(x) \leqslant P(y) + d(x, y)$ for all y. In particular, if $y \in D$, we have that $P(x) \leqslant d(x, y)$. Hence, $P(x) \leqslant \text{dist}(x, D)$.

Now, we prove that for any $\varepsilon > 0$ we have $\text{dist}(x, D) \leqslant P(x) + \varepsilon$ for all $x \in M^n$. Fix $\varepsilon > 0$. We generate a sequence (y_k) of elements of M using the first condition. Firstly, we set $y_1 = x$, any fixed element of M^n. We take y_2 to satisfy $P(y_2) \leqslant \frac{\varepsilon}{8}$ and $|P(x) - d(x, y_2)| \leqslant \frac{\varepsilon}{8}$. Inductively, we construct the rest of the sequence where y_k satisfy $P(y_k) \leqslant \frac{\varepsilon}{2^{k+1}}$ and $|P(x) - d(y_{k-1}, y_k)| \leqslant \frac{\varepsilon}{2^{k+1}}$. Therefore,

$$d(y_k, y_{k+1}) \leqslant P(y_k) + |P(x) - d(y_k, y_{k+1})| \leqslant \frac{\varepsilon}{2^k}.$$

This implies that (y_k) is a Cauchy sequence, and hence it has a limit $y \in M^n$. As P is continuous $P(y) = 0$. Moreover,

$$d(x, y) = \lim_{k\to\infty} d(y_1, y_k) \leqslant \sum_{k=1}^{\infty} d(y_k, y_{k+1}) \leqslant P(x) + \varepsilon.$$

Since $y \in D$, we have $\text{dist}(x, D) \leqslant \text{dist}(x, y) \leqslant P(x) + \varepsilon$ as required. □

2.1.2. Sets

Now we study the concept of definable sets, this concept is based on the concept of definable predicate.

Let $\mathcal{M}$ be an L-structure and let $D \subseteq M^n$ be a closed subset. We say that D is a *definable set* in $\mathcal{M}$ over A if the predicate $\text{dist}(x, D)$ is definable in $\mathcal{M}$ over A.

Lemma 2.1.9. *Let* X *be a metric space and let* $f, g : X \to [0, 1]$ *be functions such that*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X (f(x) \leqslant \delta \implies g(x) \leqslant \varepsilon).$$

Then, there exist an increasing continuous function $\alpha : [0, 1] \to [0, 1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X (g(x) \leqslant \alpha(f(x))).$$

The importance of this concept of definability for sets is that in continuous first order logic, we retain the definability of predicates if we quantify over definable sets but not over arbitrary sets.

Theorem 2.1.10. *Let* $\mathcal{M}$ *be an* L*-structure and* $D \subseteq M^n$ *a closed subset. Then, the following are equivalent:*

1. D *is definable in* $\mathcal{M}$ *over* A.
2. *For any definable predicate* $P : M^m \times M^n \to [0, 1]$ *in* $\mathcal{M}$ *over* A*, the predicate* $Q : M^m \to [0, 1]$ *defined by*

$$Q(x) = \inf\{P(x, y) : y \in D\}$$

is definable in $\mathcal{M}$ *over* A.

Proof. Assume D is definable in $\mathcal{M}$ over A. Let $P : M^m \times M^n \to [0,1]$ be any definable predicate in $\mathcal{M}$ over A. We know that P is uniformly continuous, hence by lemma 2.1.9 for $f(y,z) = |P(x,y) - P(x,z)|$ and $g(y,z) = d(y,z)$, there exists an increasing continuous function $\alpha : [0,1] \to [0,1]$ with $\alpha(0) = 0$ such that for all $x \in M^m$ and $y, z \in M^n$

$$|P(x,y) - P(x,z)| \leqslant \alpha(d(y,z)).$$

Let $Q : M^m \to [0,1]$ be $Q(x) = \inf\{P(x,y) : y \in D\}$. To prove that Q is definable we are going to show that $Q(x) = \inf\{P(x,z) + \alpha(\operatorname{dist}(z,D)) : z \in M^n\}$. Hence if P and $\operatorname{dist}(z,D)$ are definable in $\mathcal{M}$ over A, so is Q by lemma 2.1.1. Notice that the infimum is taken over M^n, so it is expressible in continuous logic.

We have $P(x,y) \leqslant P(x,z) + \alpha(d(y,z))$ for all $x \in M^m$ and $y, z \in M^n$ due to our choice of α. Taking the infimum over $y \in D$ and noting that since α is continuous and increasing it satisfies $\inf_{y \in D} \alpha(d(z,y)) = \alpha(\operatorname{dist}(z,D))$, we get that

$$Q(x) \leqslant P(x,z) + \alpha(\operatorname{dist}(z,D))$$

for all $x \in M^m$ and $z \in M^n$. Finally, we have

$$Q(x) \leqslant \inf\{P(x,z)+\alpha(\operatorname{dist}(z,D)) : z \in M^n\} \leqslant \inf\{P(x,z)+\alpha(\operatorname{dist}(z,D)) : z \in D\} = Q(x)$$

for all $x \in M^m$.

For the converse, we set $m = n$ and $P(x,y) = d(x,y)$. This implies

$$Q(x) = \inf\{P(x,y) : y \in D\} = \operatorname{dist}(x,D)$$

is definable in $\mathcal{M}$ over A. Hence D is definable in $\mathcal{M}$ over A. □

The following result shows some properties that definable sets have with respect to elementary substructures.

Proposition 2.1.11. *Let $\mathcal{N}, \mathcal{M}$ be substructures such that $\mathcal{N} \preccurlyeq \mathcal{M}$ and let $D \subseteq M^n$ be definable in $\mathcal{M}$ over A, with $A \subseteq N$. Then, the following are satisfied:*

1. $\operatorname{dist}(x,D) = \operatorname{dist}(x, D \cap N^n)$ *for all $x \in N^n$. Hence, $D \cap N^n$ is definible in $\mathcal{N}$ over A.*
2. $(\mathcal{N}, \operatorname{dist}(\cdot, D \cap N^n)) \preccurlyeq (\mathcal{M}, \operatorname{dist}(\cdot, D))$.
3. *If $D \neq \varnothing$ then $D \cap N^n \neq \varnothing$.*

Proof. (1): Consider the function $P : N^n \to [0,1]$ defined by $P(x) = \operatorname{dist}(x,D)$, the zeroset of P is $D \cap N^n$ and P satisfy the conditions in theorem 2.1.8. Hence, this gives us $\operatorname{dist}(x,D) = \operatorname{dist}(x, D \cap N^n)$ for any $x \in N^n$ and so $D \cap N^n$ is definable in $\mathcal{N}$ over A.

Statement (2) follows from (1) by proposition 2.1.3.

To prove statement (3), note that if $D \neq \varnothing$, then $\inf_x \operatorname{dist}(x,D) = 0$. Hence, by statement (2), $\inf_x \operatorname{dist}(x, D \cap N^n) = 0$ in $\mathcal{N}$. This implies that there exists $a \in N^n$ such that $\operatorname{dist}(a, D \cap N^n) < 1$. Therefore $D \cap N^n \neq \varnothing$. □

If a closed $D \subseteq M^n$ is a definable set, then it is the set of zeroes of a definable predicate, but the converse is not true in general. The next result tries to illustrate the difference between definable sets and zerosets of definable predicates.

Proposition 2.1.12. *Let $\mathcal{M}$ be an L-structure and let $D \subseteq M^n$ be a closed subset. Then the following are equivalent:*

(1) *D is definable in $\mathcal{M}$ over A.*

(2) *There exists a sequence of $L(A)$-formulas $(\varphi_k : k \geqslant 1)$ and a sequence of positive real numbers $(\delta_k : k \geqslant 1)$ such that all $x \in D$ satisfy $\varphi_k^{\mathcal{M}}(x) = 0$ for all $k \geqslant 1$ and*
$$\varphi_k^{\mathcal{M}}(x) \leqslant \delta_k \implies \mathrm{dist}(x, D) \leqslant \frac{1}{k}.$$
for all $x \in M^n$ and $k \geqslant 1$.

(3) *There exist an n-ary predicate P definable in $\mathcal{M}$ over A such that all $x \in D$ satisfy $P(x) = 0$ and*
$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in M^n (P(x) \leqslant \delta \implies \mathrm{dist}(x, D) \leqslant \varepsilon).$$

Proof. (1) $\implies$ (2): Assume $Q(x) = \mathrm{dist}(x, D)$ is definable in $\mathcal{M}$ over A. So there exists a sequence $(\psi_m : m \geqslant 1)$ of $L(A)$-formulas such that for all $x \in M^n$ and all $m \geqslant 1$ we have
$$|Q(x) - \psi_m^{\mathcal{M}}(x)| \leqslant \frac{1}{3m}.$$
Hence if $x \in D$, $\psi_m^{\mathcal{M}}(x) \leqslant \frac{1}{3m}$. Also, by triangular inequality, if $\psi_m^{\mathcal{M}}(x) \leqslant \frac{2}{3m}$ we have
$$Q(x) \leqslant \psi_m^{\mathcal{M}}(x) + |Q(x) - \psi_m^{\mathcal{M}}(x)| \leqslant \frac{1}{m}.$$
Hence the $L(A)$-formulas $\varphi_m(x) := \psi_m(x) \dot{-} \frac{1}{3m}$ have the required properties with $\delta_m \leqslant \frac{1}{3m}$.

(2) $\implies$ (3): The predicate $P(x) = \sum_{m=1}^{\infty} 2^{-m} \varphi_m^{\mathcal{M}}(x)$ is definable in $\mathcal{M}$ over A since the partial sums converge to $P(x)$. The predicate P satisfies the required properties since $\varphi_m^{\mathcal{M}}(x) = 0$ for all $x \in D$ and all $m \geqslant 1$ and for every $\varepsilon > 0$, we can find $\delta > 0$ such that $P(x) \leqslant \delta$ implies $\varphi_m^{\mathcal{M}}(x)\delta_m$ for an m big enough to satisfy $\frac{1}{m} \leqslant \varepsilon$.

(3) $\implies$ (1): Proposition 2.1.9 gives us a continuous increasing function $\alpha : [0, 1] \to [0, 1]$ such that $\alpha(0) = 0$ and for all $x \in M^n$, $\mathrm{dist}(x, D) \leqslant \alpha(P(x))$. Let F be the predicate defined by
$$F(x) = \inf_y \min(\alpha(P(y)) + d(x, y), 1).$$
As P is definable in $\mathcal{M}$ over A, so is F since α is a connective. Notice that we have $F(x) \leqslant \mathrm{dist}(x, D)$ because $P(y) = 0$ if $y \in D$. On the other hand,for all $y \in M^n$, we have $\mathrm{dist}(y, D) \leqslant \alpha(P(y))$ and hence
$$F(x) \geqslant \inf_y \min(\alpha(P(y)) + d(x, y), 1) \geqslant \min(\mathrm{dist}(y, D), 1) = \mathrm{dist}(y, D).$$
Therefore D is definable in $\mathcal{M}$ over A. □

Finally, we introduce the concept of definable functions.

2.1.3. Functions

Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. We say that the function f is definable in $\mathcal{M}$ over A if the predicate $d(f(x), y)$ is definable in $\mathcal{M}$ over A.

Proposition 2.1.13. *Let f be a definable function in $\mathcal{M}$ over A, then f is uniformly continuous.*

Proof. We already know definable predicates are uniformly continuous functions. Let $\Delta : (0,1] \to (0,1]$ be a modulus of uniform continuity for the definable predicate $d(f(x), y)$. We have that if $d(x, x') < \Delta(\varepsilon)$ and $d(y, y') < \Delta(\varepsilon)$, then

$$|d(f(x), y) - d(f(x'), y')| \leqslant \varepsilon.$$

Taking $y = y' = f(x')$ we get that $d(x, x') < \Delta(\varepsilon)$ implies $d(f(x), f(x')) \leqslant \varepsilon$. □

Proposition 2.1.14. *Let κ be an uncountable cardinal, $\mathcal{M}$ be a κ-saturated structure and $A \subseteq M$ such that $|A| < \kappa$. Let $f : M^n \to M$ be any function, the following are equivalent:*

(1) *f is definable in $\mathcal{M}$ over A.*

(2) *$\mathcal{G}_f$, the graph of f, is type-definable in $\mathcal{M}$ over A.*

Proof. (1) $\Longrightarrow$ (2) We claim that if $f : M^n \to M$ is a definable function in $\mathcal{M}$ over A, then $\mathcal{G}_f$ is a definable set in $\mathcal{M}$ over A. To prove the claim, note that since f is a definable function in $\mathcal{M}$ over A, the predicate $d(f(z), y)$ is definable in $\mathcal{M}$ over A. Then, the claim follows from the following equality

$$\operatorname{dist}((x, y), \mathcal{G}_f) = \inf_z \max(d(x, z), d(f(z), y)).$$

Note that the saturation hypothesis is not needed.

To prove (2) $\Longrightarrow$ (1) we make use of corollary 2.1.6. Hence, we need to check that for the predicate $P : M^{n+1} \to [0,1]$ defined by $P(x, y) = d(f(x), y)$ for all $x \in M^n$, $y \in M$ the sets $\{(a, b) \in M^{n+1} : P(a, b) \leqslant r\}$ and $\{(a, b) \in M^{n+1} : P(a, b) \geqslant r\}$ are type-definable in $\mathcal{M}$ over A for all $r \in [0,1]$. To do so, let $\Gamma(x, y)$ be the set of $L(A)$-conditions that type-defines $\mathcal{G}_f$ in $\mathcal{M}$. Note that for a fixed $r \in [0,1]$ we have

$$P(x, y) \leqslant r \iff \exists z((x, z) \in \mathcal{G}_f \wedge d(z, y) \leqslant r)$$

and

$$P(x, y) \geqslant r \iff \exists z((x, z) \in \mathcal{G}_f \wedge d(z, y) \geqslant r).$$

Hence the set $\{(a, b) \in M^{n+1} : P(a, b) \leqslant r\}$ is type-defined in $\mathcal{M}$ by the set of $L(A)$-conditions of the form

$$\inf_z \max(\varphi(x, z), d(z, y)) = 0,$$

where $\varphi = 0$ is any condition in Γ. The same argument applies to the set $\{(a, b) \in M^{n+1} : P(a, b) \geqslant r\}$, as $r \in [0,1]$ was arbitrary, we can apply corollary 2.1.6. □

Proposition 2.1.15. *Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Suppose that the function $f : M^n \to M$ is definable in $\mathcal{M}$ over A. Then:*

(1) *If $\mathcal{N} \preccurlyeq \mathcal{M}$ and $A \subseteq N$, then f maps N^n into N and the restriction of f to N^n is definable in $\mathcal{N}$ over A.*

(2) *If $\mathcal{M} \preccurlyeq \mathcal{N}$ then there is a function $g : N^n \to N$ such that g extends f and g is definable in $\mathcal{N}$ over A.*

Proof. (1) We show that f maps N^n to N, so that the restriction of f on $\mathcal{N}$ makes sense. The definability of f in $\mathcal{N}$ over A will follow immediately. Fix any element $(a_1, \dots, a_n) \in N^n$ and let P be the predicate defined by $P(y) = d(f(a_1, \dots, a_n), y)$ for all $y \in M$. This predicate is definable in $\mathcal{M}$ over $A \cup \{a_1, \dots, a_n\} \subseteq N$. Let the predicate $Q : N \to [0, 1]$ be the restriction of P to N. The same sequence of formulas whose evaluation converge to P witness that Q is a definable predicate in $\mathcal{N}$ over $A \cup \{a_1, \dots, a_n\}$. Then, applying proposition 2.1.3 we get $(\mathcal{N}, Q) \preccurlyeq (\mathcal{M}, P)$. Since, for all $x, y \in M^n$ we have $\inf_y P(y) = 0$ and $d(x, y) \leqslant P(x) + P(y)$, $(\mathcal{N}, Q) \preccurlyeq (\mathcal{M}, P)$ implies that Q satisfies $\inf_y Q(y) = 0$ and $d(x, y) = Q(x) + Q(y)$ for all $x, y \in N^n$. We can use this properties to construct a sequence $(c_k : k \geqslant 1)$ of elements of N that satisfy $Q(c_k) \leqslant \frac{1}{k}$ and $d(c_k, c_l) \leqslant \frac{1}{k} + \frac{1}{l}$. This means that $(c_k : k \geqslant 1)$ is a convergent sequence in N, since N is complete $b = \lim_{k\to\infty} c_k$ is in N and $P(b) = Q(b) = 0$. However, the only zero of P is $f(a_1, \dots, a_n)$, hence $f(a_1, \dots, a_n) = b \in N$.

(2) Let $\mathcal{M} \preccurlyeq \mathcal{N}$. The statement (1) allows us to assume that $\mathcal{N}$ is ω_1-saturated without loss of generality. Let $P(x, y)$ be $d(f(x), y)$, by proposition 2.1.4, there exists $Q : N^{n+1} \to [0, 1]$ a definable predicate in $\mathcal{N}$ over A extending P and satisfying $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$. Since f is a function we have $\sup_x \inf_y P(x, y) = 0$ and hence the same is true for Q.

We want to define the extension g via Q. We first claim that for all $x \in N^n$ there exists at least one $y \in N$ such that $Q(x, y) = 0$. To prove the claim, note first that if $(\varphi_k : k \geqslant 1)$ is the sequence of functions whose interpretations in $\mathcal{N}$ converge uniformly to Q, then $\sup_x \inf_y Q(x, y) = \sup_x \lim\limits_{k\to\infty} (\inf_y \varphi_k^{\mathcal{N}}(x, y))$. Now, let $x \in N^n$ be a fixed element the statement. We claim that the statement $\lim\limits_{k\to\infty} (\inf_y \varphi_k^{\mathcal{N}}(x, y)) = 0$ is equivalent to a finitely satisfiable set of $L(\widetilde{A})$-conditions, where $\widetilde{A}$ is the set of elements of A that appears in at least one of the formulas φ_k. Let $(c_k : k \geqslant 1)$ be the sequence $(\inf_y \varphi_k^{\mathcal{N}}(x, y) : k \geqslant 1)$. Without loss of generality, we can assume that $(c_k : k \geqslant 1)$ is a non increasing sequence of real numbers tending to 0. From the fact that $(\varphi_k^{\mathcal{N}} : k \geqslant 1)$ is uniformly convergent, we get that for any $j \in \mathbb{N}$ there exist $\varepsilon_j > 0$ such that if $k, l \geqslant j$, then $|\varphi_k - \varphi_l| \leqslant \varepsilon_j$. Hence, the set $\Gamma(y) = \{\varphi^{\mathcal{N}}(x, y) \leqslant c_k + \varepsilon_k : k \geqslant 1\}$ is finitely satisfiable. Using now the ω_1-saturation of $\mathcal{N}$, we get that for each element $x \in N^n$ there exists at least one $y \in N$ satisfying all the conditions in Γ, this implies $Q(x, y) = \lim\limits_{k\to\infty} \varphi_k^{\mathcal{N}}(x, y) = 0$, as we claimed.

Now, to prove the uniqueness of the element y satisfying $Q(x, y) = 0$ for each $x \in N^n$, note that from the definition of P and the triangle inequality follows that P

satisfies

$$\sup_x \sup_y \sup_{y'} (|d(y,y') - P(x,y')| \dot{-} P(x,y)) = 0.$$

Since the same is true for Q, it follows that for each $x \in N^n$ there is at most one $y \in N^n$ such that $Q(x,y) = 0$. Therefore, the zeroset of Q is the graph of some function. It also follows that if $Q(x,y) = 0$ then $Q(x,y') = d(y,y')$ for all $y' \in N$. So we can define the function g via $Q(x,y') = d(g(x),y')$ for all $y' \in N$. Hence g is definable in $\mathcal{N}$ over A as required. □

Corollary 2.1.16. *Let $f, g, f_1, \dots, f_n$ be definable functions in $\mathcal{M}$ over A and P be a definable predicate in $\mathcal{M}$ over A. Then, $f \circ g$ is a definable function in $\mathcal{M}$ over A and $P(f_1, \dots, f_n)$ is a definable predicate in $\mathcal{M}$ over A.*

Proof. Using proposition 2.1.4 and proposition 2.1.15 we may work in a sufficiently saturated extension $\mathcal{M} \preccurlyeq \mathcal{N}$.

Let $f : N^n \to N$ and $g : N \to N$ be two definable functions in $\mathcal{N}$ over A. First, we are going to show that $\mathcal{G}_{g \circ f}$ is type-definable. We have

$$(x,y) \in \mathcal{G}_{g\circ f} \iff \exists z((x,z) \in \mathcal{G}_f \wedge (z,y) \in \mathcal{G}_g).$$

If Γ_f and Γ_g are the sets of $L(A)$-formulas witnessing the type-definability of $\mathcal{G}_f$ and $\mathcal{G}_g$ respectively, that is the conditions that type-define $\mathcal{G}_f$ are of the form $\varphi = 0$ with $\varphi \in \Gamma_f$ and the same for Γ_g. Then $\mathcal{G}_{g\circ f}$ is type-defined in $\mathcal{N}$ over A by the set of conditions

$$\inf_z \max(\varphi(x,z), \psi(z,y)) = 0,$$

where φ is a formula from Γ_f and ψ is a formula from Γ_g. Hence, by proposition 2.1.14, $g \circ f$ is a definable function in $\mathcal{N}$ over A.

For the second statement, we do the case of a definable predicate and two definable functions, one can see that this proof is easy to generalize to the case of n definable functions.

Let $P : N^2 \to [0,1]$ be a definable predicate and let $f : N^n \to N$, $g : N^m \to N$ be definable functions. We claim that the predicate $Q : N^{n+m} \to [0,1]$ defined by $Q(x,y) = P(f(x), g(y))$ is definable in $\mathcal{N}$ over A. To prove it, we are going to check that for any $r \in [0,1]$ the sets $\{(x,y) \in N^{n+m} : Q(x,y) \leqslant r\}$ and $\{(x,y) \in N^{n+m} : Q(x,y) \geqslant r\}$ are type-definable in $\mathcal{N}$ over A. We have that

$$Q(x,y) \leqslant r \iff \exists z_1 \exists z_2((x,z_1) \in \mathcal{G}_f \wedge (x,z_2) \in \mathcal{G}_g \wedge P(z_1,z_2) \leqslant r).$$

Since the sets $\mathcal{G}_f$, $\mathcal{G}_g$ and $\{(z_1,z_2) \in N^2 : P(z_1,z_2) \leqslant r\}$ are type-definable in $\mathcal{N}$ over A, let Γ_f, Γ_g and Γ_P be the sets of $L(A)$-formulas witnessing the type-definability of those sets. Then $\{(x,y) \in N^{n+m} : Q(x,y) \leqslant r\}$ is type-defined in $\mathcal{N}$ over A by the set of conditions

$$\inf_{z_1} \inf_{z_2} \max(\varphi(x,z_1), \psi(y,z_2), \sigma(z_1,z_2)) = 0,$$

where φ is a formula from Γ_f, ψ is a formula from Γ_g and σ is a maximum of a finite set of formulas from Γ_P.An analogous argument applies to $\{(x,y) \in N^{n+m} : Q(x,y) \geqslant r\}$ and hence $Q(x,y)$ is a definable predicate in $\mathcal{N}$ over A. □

2.2. Algebraic and definable elements

Let $\mathcal{M}$ be an L-structure, $A \subseteq M$ a subset of M and $a \in M^n$ a tuple of M. We say that a is *definable* in $\mathcal{M}$ over A if the set $\{a\}$ is definable in $\mathcal{M}$ over A. We say that a is *algebraic* in $\mathcal{M}$ over A if there exists a compact set $C \subseteq M^n$ such that $a \in C$ and C is definable in $\mathcal{M}$ over A. The set of all definable points in $\mathcal{M}$ over A is called the *definable closure* of A in $\mathcal{M}$ and it is denoted by $dcl_{\mathcal{M}}(A)$. The set of all algebraic points in $\mathcal{M}$ over A is called the *algebraic closure* of A in $\mathcal{M}$ and it is denoted by $acl_{\mathcal{M}}(A)$.

As in classical first order model theory, the properties of the tuples reduce to the properties of their coordinates.

Proposition 2.2.1. *Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Let $a \in M^n$ be any tuple. Then, a is definable in $\mathcal{M}$ over A if and only if a_i is definable in $\mathcal{M}$ over A for each $i = 1, \dots, n$. The same is true if we substitute algebraic for definable in the previous statement.*

Proof. We begin with the definable case. Suppose that $a \in M^n$ is definable in $\mathcal{M}$ over A. Note that the predicates $Q_i : M^{2n} \to [0,1]$ defined by $Q_i(x,y) = d(x_i, y_i)$ are definable predicates for $i = 1, \dots, n$. Hence, applying theorem 2.1.10, the predicates $P_i : M^n \to [0,1]$ defined by $P_i(x) = d(x_i, a_i) = P_i(x_i, \dots, x_i)$ are definable for all $i = 1, \dots, n$. Thus, a_i is definable in $\mathcal{M}$ over A for all $i = 1, \dots, n$. Conversely, assume that a_i is definable in $\mathcal{M}$ over A for all $i = 1, \dots, n$. Then, we have that

$$d(x,a) = \max(d(x_1,a_1), \dots, d(x_n,a_n)).$$

Hence, $a \in M^n$ is definable in $\mathcal{M}$ over A by lemma 2.1.1.

Now, for the algebraic case, we follow the same strategy. Suppose that $a \in M^n$ is definable in $\mathcal{M}$ over A and C is the compact set witnessing this property. Note that the projection C_i over the i-th coordinate is a compact set containing a_i. Hence, it suffices to prove that C_i is definable for each $i = 1, \dots, n$. As in the definable case, note that the predicates $Q_i : M^{2n} \to [0,1]$ defined by $Q_i(x,y) = d(x_i, y_i)$ are definable predicates for $i = 1, \dots, n$. Hence, applying theorem 2.1.10, the predicates $P_i : M^n \to [0,1]$ defined by $P_i(x) = \operatorname{dist}(x_i, C_i) = P_i(x_i, \dots, x_i)$ are definable for all $i = 1, \dots, n$. Therefore, the sets C_i for $i = 1, \dots, n$ are definable in $\mathcal{M}$ over A. For the converse, assume that $C_1, \dots, C_n$ are compact subsets of M witnessing that $a_1, \dots, a_n$ are algebraic in $\mathcal{M}$ over A. The product $C = C_1 \times \dots \times C_n$ is a compact subset of M^n containing a. We claim that C is also a definable subset of M^n. To prove the claim, note that

$$\operatorname{dist}((x_1, \dots, x_n), C) = \inf_{y_1 \in C_1} \dots \inf_{y_n \in C_n} \max(d(x_1,y_1), \dots, d(x_n,y_n))$$

and the right side is definable by lemma 2.1.1 and theorem 2.1.10. □

The definable and algebraic closures depend only on A and not on the structure in which they are defined.

Proposition 2.2.2. *Let $\mathcal{M} \preccurlyeq \mathcal{N}$ and $A \subseteq M$. If $C \subseteq N^n$ is definable in $\mathcal{N}$ over A and $C \cap M^n$ is compact, then $C \subseteq M^n$.*

Proof. Assume that the predicate $Q(x) = \mathrm{dist}(x, C)$ is definable in $\mathcal{M}$ over A and let P be the restriction of Q to M^n. Proposition 2.1.11 implies that $P(x) = \mathrm{dist}(x, C \cap M^n)$ for all $x \in M^n$ and proposition 2.1.3 implies $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$. Since $C \cap M^n$ is compact, there exists a finite ε-net for each $\varepsilon > 0$. Fix $\varepsilon > 0$ and let $c_1, \dots c_m$ be a finite ε-net in $C \cap M^n$. This implies that if $P(x) < \varepsilon$, then $d(x, c_j) \leqslant 2\varepsilon$ for some $j = 1, \dots, m$. We can write the last statement as a closed condition

$$\sup_x \min(\varepsilon \dot{-} P(x), \min(d(x, c_1), \dots, d(x, c_m) \dot{-} 2\varepsilon)) = 0,$$

this condition holds in $(\mathcal{M}, P)$, hence the condition

$$\sup_x \min(\varepsilon \dot{-} P(x), \min(d(x, c_1), \dots, d(x, c_m) \dot{-} 2\varepsilon)) = 0$$

holds in $(\mathcal{N}, Q)$. It follows that $c_1, \dots, c_m$ is a finite 2ε-net in C. Letting ε tend to 0, we see that every element in C is the limit of a sequence of elements in M^n, since M^n is complete, this implies that $C \subseteq M^n$ as required. □

Corollary 2.2.3. *For L-structures $\mathcal{M}$, $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$ we have the following chain of inclusions*

$$dcl_{\mathcal{M}}(A) = dcl_{\mathcal{N}}(A) \subseteq acl_{\mathcal{N}}(A) = acl_{\mathcal{M}}(A).$$

Proof. It is clear that for any L-structure $\mathcal{N}$, $dcl_{\mathcal{N}}(A) \subseteq acl_{\mathcal{N}}(A)$ since singletons are compact sets.

First, we are going to show that $dcl_{\mathcal{M}}(A) \subseteq dcl_{\mathcal{N}}(A)$ and $acl_{\mathcal{M}}(A) \subseteq acl_{\mathcal{N}}(A)$. To do so, suppose that $C \subseteq M^n$ is compact and definable in $\mathcal{M}$ over A. This means that the predicate $P(x) = \mathrm{dist}(x, C)$ is definable in $\mathcal{M}$ over A and hence by proposition 2.1.4, there exist a predicate $Q : N^n \to [0, 1]$ definable in $\mathcal{N}$ over A extending P such that $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$. Let $D \subseteq N^n$ be the zeroset of Q. It is easy to check that P satisfies the conditions in theorem 2.1.8, and hence, so does Q. Thus, Q is the predicate $\mathrm{dist}(x, D)$ for all $x \in N^n$. Therefore, by proposition 2.2.2, it follows that $D = C$. This implies that C is definable in $\mathcal{N}$ over A and hence $dcl_{\mathcal{M}}(A) \subseteq dcl_{\mathcal{N}}(A)$ and $acl_{\mathcal{M}}(A) \subseteq acl_{\mathcal{N}}(A)$.

For the converse, note that if $C \subseteq N^n$ is a compact set, we have that $C \cap M^n$ is also a compact set. Hence if C is also a definable set in $\mathcal{N}$ over A, by proposition 2.2.2 we have $C \subseteq M^n$. This implies that C is compact and definable in $\mathcal{M}$ over A. Moreover, this shows $dcl_{\mathcal{N}}(A) \subseteq dcl_{\mathcal{M}}(A)$ and $acl_{\mathcal{N}}(A) \subseteq acl_{\mathcal{M}}(A)$ □

The previous proof shows the following corollary.

Corollary 2.2.4. *If $C \subseteq M^n$ is compact and definable in $\mathcal{M}$ over A, then, it is compact and definable over A in any elementary extension of $\mathcal{M}$.*

As long as we work in sufficiently saturated models, we can define the algebraic and definable closures in terms of compact zerosets instead of compact definable sets.

Proposition 2.2.5. *Let $\mathcal{M}$ be an ω_1-saturated L-structure and $A \subseteq M$. If C is a compact subset of M^n, the following are equivalent:*

(1) *C is definable in $\mathcal{M}$ over A.*

(2) *C is the zeroset of a definable predicate in $\mathcal{M}$ over A.*

Proof. Let $P : M^n \to [0,1]$ be a definable predicate in $\mathcal{M}$ over A whose zeroset is C. Giver $\varepsilon > 0$, let $F \subseteq C$ be a finite $\frac{\varepsilon}{2}$-net in C. We claim that there exists $\delta > 0$ such that any a satisfying $P(a) \leqslant \delta$ must be at distance within ε of some element of F. To prove the claim, suppose that it is false. Hence, for any $k \geqslant 1$ we can find an element b satisfying $P(b) \leqslant \frac{1}{k}$ and $d(b,c) \geqslant \varepsilon$ for all $c \in F$. Then, the ω_1-saturation of $\mathcal{M}$ gives us an element $a \in M^n$ satisfying $P(a) \leqslant \frac{1}{k}$ for every $k \geqslant 1$ and $d(a,c) \geqslant \varepsilon$ for all $c \in F$ which is a contradiction. The existence of such $\delta > 0$, for each $\varepsilon > 0$, implies that P satisfies the conditions in proposition 2.1.12(3). Hence C is definable in $\mathcal{M}$ over A.

The converse is trivial since C is the zeroset of the definable predicate $\mathrm{dist}(x, C)$. □

Proposition 2.2.6. *Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, and $a \in M^n$. The following statements are equivalent:*

(1) *a is definable in $\mathcal{M}$ over A.*

(2) *For any $\mathcal{N}$ elementary extension of $\mathcal{M}$ the only realization of $\mathrm{tp}_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is a.*

(3) *For any $\varepsilon > 0$ there is an $L(A)$-formula $\varphi(x)$ and $\delta > 0$ such that $\varphi^{\mathcal{M}}(a) = 0$ and the diameter of $\{b \in M^n : \varphi^{\mathcal{M}}(b) \leqslant \delta\}$ is less than ε.*

Proof. (1) $\Longrightarrow$ (2). Let Q be the predicate definable in $\mathcal{N}$ over A extending $d(x,a)$ such that $(\mathcal{M}, d(x,a)) \preccurlyeq (\mathcal{N}, Q)$. Since $d(x,a)$ satisfy

$$\sup_x \sup_{x'} (|d(x',x) - d(x',)a| \dot{-} d(x,a)) = 0,$$

the same is true for $Q(x)$. This condition implies that Q has at most one zero in $\mathcal{N}$.

(2) $\Longrightarrow$ (3). We may assume that $\mathcal{M}$ is κ-saturated with $\kappa > |A|$ due to corollary 2.2.3. Suppose that (3) does not hold. That is, there exists $\varepsilon > 0$ such that for any $k \geqslant 1$ and any $\varphi \in \mathrm{tp}(a/A)$ the set $\{b \in M^n : \varphi^{\mathcal{M}}(b) \leqslant \frac{1}{k}\}$ has diameter grater than ε. That is, the set of $L(A)$-conditions

$$\mathrm{tp}(x/A)^+ \cup \mathrm{tp}(y/A)^+ \cup d(x,y) > \varepsilon$$

is finitely satisfiable. Hence, the saturation of $\mathcal{M}$ implies then that there exist elements $c, d \in M^n$ such that c and d satisfy $\mathrm{tp}(a/A)$ and $d(c,d) > \varepsilon$. This contradicts (2).

(3) $\Longrightarrow$ (1). Follows from proposition 2.1.12. □

We introduce some characterizations on being algebraic.

Lemma 2.2.7. *Let $\mathcal{M}$ be an L-structure, $A \subseteq M$ and let $a \in M^n$. Then, $a \in acl_{\mathcal{M}}(A)$ if and only if there exists some predicate P definable in $\mathcal{M}$ over A such that $P(a) = 0$ and $\{b \in N : Q(b) = 0\}$ is compact for all $\mathcal{N}$ and Q satisfying $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$.*

Proof. Suppose that a is algebraic in $\mathcal{M}$ over A. This implies that there exists a compact set $C \subseteq M^n$ definable in $\mathcal{M}$ over A with $a \in C$ and so $\mathrm{dist}(x, C)$ is a definable predicate in $\mathcal{M}$ over A. By corollary 2.2.4 and the uniqueness of the predicate extending $\mathrm{dist}(x, C)$, the zeroset of Q is C and hence is compact.

Now, suppose that there exists $P : M^n \to [0, 1]$ definable in $\mathcal{M}$ over A such that $P(a) = 0$ and for all $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$, the set $C_{\mathcal{N}} = \{b \in N^n : Q(b) = 0\}$ is compact. In particular, let $\mathcal{N}$ to be ω_1-saturated. Then, by proposition 2.2.5, $C_{\mathcal{N}}$ is definable in $\mathcal{N}$ over A. Finally, as $C_{\mathcal{N}} \cap M^n$ is compact, proposition 2.2.2 implies that $C = C_{\mathcal{N}} \cap M^n$ is a compact set definable in $\mathcal{M}$ over A, and contains a. □

Actually, the previous proof shows a stronger property.

Corollary 2.2.8. *Let $\mathcal{M}$ be an L-structure, $A \subseteq M$ and let $a \in M^n$. Then, $a \in acl_{\mathcal{M}}(A)$ if and only if there exists some predicate P definable in $\mathcal{M}$ over A such that $P(a) = 0$ and $\{b \in N : Q(u) = 0\}$ is compact for some ω_1-saturated L-structure $\mathcal{N}$ and some predicate Q satisfying $(\mathcal{M}, P) \preccurlyeq (\mathcal{N}, Q)$.*

Now, we show a sufficient condition for being algebraic that will be useful in later proofs.

Lemma 2.2.9. *Let $\mathcal{M}$ be an ω_1-saturated L-structure with $A \subseteq M$ and $a \in M$. If for every $n \geqslant 1$ there exists an $L(A)$-formula φ_n such that $\varphi_n^{\mathcal{M}}(a) = 0$ and the zeroset of φ_n has a finite $\frac{1}{n}$-net, then a is algebraic in $\mathcal{M}$ over A.*

Proof. Let C_n be the zeroset of φ_n in $\mathcal{M}$, the set $C = \bigcap_{n=1}^{\infty} C_n$ is the zeroset of the predicate $P = \sum_{n \geqslant 1} 2^{-n} \varphi_n$ in $\mathcal{M}$. The set C has a finite $\frac{1}{n}$-net for every $n \geqslant 1$ since we can modify the original net on C_n to be a $\frac{1}{n}$-net by adding finitely many new points, hence, C is a compact set. Proposition 2.2.5 then implies that the set C is definable in $\mathcal{M}$ over A and contains a. □

Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. The *bounded closure* of A in $\mathcal{M}$, denoted $bdd_{\mathcal{M}}(A)$, is the set of elements $a \in M^n$ for which there exists some cardinal τ such that for any elementary extension $\mathcal{N}$ of $\mathcal{M}$, the set of realizations of $\mathrm{tp}(a/A)$ in $\mathcal{N}$ has cardinality less than τ.

Theorem 2.2.10. *Let $\mathcal{M}$ be an L-structure with $A \subseteq M$. Then, $acl_{\mathcal{M}}(A) = bdd_{\mathcal{M}}(A)$*

Proof. Let $a \in acl_{\mathcal{M}}(A)$ and let P be as in lemma 2.2.7. Let $(\mathcal{N}, Q) \succcurlyeq (\mathcal{M}, P)$ and let S be the set of realizations of $\mathrm{tp}(a/A)$ in $\mathcal{N}$. It is clear that $S \subseteq \{b \in N : Q(b) = 0\}$. Since the set $\{b \in N : Q(b) = 0\}$ is compact, there exists a countable dense subset.

This implies that any element of S can be identified with the limit of a sequence of elements in the dense subset. Hence, the following holds

$$|S| \leqslant |\{b \in N : Q(b) = 0\}| \leqslant 2^{\aleph_0}.$$

This implies that $a \in bdd_{\mathcal{M}}(A)$.

Now, we may assume $\mathcal{M}$ is ω_1-saturated. Let $a \in M \backslash acl_{\mathcal{M}}(A)$. The auxiliary lemma 2.2.9 implies that there exists some $n \geqslant 1$ such that given any cardinal τ the set of conditions

$$\Sigma = \{0 = \varphi(x_\alpha) : \alpha < \tau; \varphi^{\mathcal{M}}(a) = 0\} \cup \{0 = \frac{1}{n} \dot{-} d(x_\alpha, x_\beta) : \alpha < \beta < \tau\}$$

is finitely satisfiable in $\mathcal{M}$. Let $\mathcal{M}'$ be a κ-saturated elementary extension of $\mathcal{M}$ with $\kappa > \tau$. Then Σ is realized in $\mathcal{M}'$ by $(a_\alpha : \alpha < \tau)$. However, since any a_α is a realization of $\text{tp}(a/A)$ in $\mathcal{M}$, the set of realizations of $\text{tp}(a/A)$ in $\mathcal{M}$ has cardinality grater than τ. hence $a \notin bdd_{\mathcal{M}}(A)$. □

Corollary 2.2.11. *Let $\mathcal{M}$ be a κ saturated L-structure with $\kappa > 2^{\aleph_0}$. Let S be the set of realizations of any type $\text{tp}(a/A)$ with $|A| < \kappa$. Then, either $|S| \leqslant 2^{\aleph_0}$ and so every element of S is algebraic, or $|S| \geqslant \kappa$.*

We prove some properties of the algebraic closure.

Proposition 2.2.12. *For any L-structure $\mathcal{M}$ such that $A, B \subseteq M$ the following statements hold:*

(1) *$A \subseteq acl_{\mathcal{M}}(A)$;*

(2) *if $A \subseteq acl_{\mathcal{M}}(B)$ then $acl_{\mathcal{M}}(A) \subseteq acl_{\mathcal{M}}(B)$;*

(3) *if $a \in acl_{\mathcal{M}}(A)$ then there exists a countable $A_0 \subseteq A$ such that $a \in acl_{\mathcal{M}}(A_0)$;*

(4) *if A is a dense subset of B, then $acl_{\mathcal{M}}(A) = acl_{\mathcal{M}}(B)$.*

Proof. (1) For any $a \in A$, the set $\{a\}$ is compact and $d(x, a)$ is definable in $\mathcal{M}$ over A.

(2) We may assume that $\mathcal{M}$ is strongly κ-homogeneous, with $\kappa > |B|$. Let a be an element of $acl_{\mathcal{M}}(A)$. The homogeneity of $\mathcal{M}$ implies that for any $b \in M^n$ satisfying $\text{tp}(a/B)$ there exists $\sigma \in Aut_B(\mathcal{M})$ with $\sigma(a) = b$. We fix an isomorphism σ_b for each b satisfying $\text{tp}(a/B)$. Let S be the set of realizations of $\text{tp}(a/B)$ in $\mathcal{M}$. We define the following equivalence relation in S: $b_1 \sim b_2$ if $\sigma_{b_1}(x) = \sigma_{b_2}(x)$ for all $x \in A$. Note that if $b_1 \sim b_2$ then $\text{tp}(b_1/\sigma_{b_1}(A)) = \text{tp}(b_2/\sigma_{b_2}(A))$. Hence, $|S/\sim|$ is less than the number of possible images of A under B-isomorphism. However, since every element of A is algebraic over B, by corollary 2.2.11, every element can only have $2^{\aleph_0}$ images, hence $|S/\sim| \leqslant (2^{\aleph_0})^{|A|}$. Note also that for any given $b \in S$, the equivalence class $[b]_\sim$ is a subset of the realizations of $\text{tp}(\sigma_b(a)/\sigma_b(A))$. Since the latter set is in bijection with the set of realizations of $\text{tp}(a/A)$, and a is algebraic over A, corollary 2.2.11 implies that $|[b]_\sim| \leqslant 2^{\aleph_0}$. Thus $|S| \leqslant (2^{\aleph_0})^{|A|} 2^{\aleph_0} = (2^{\aleph_0})^{|A|}$. Since S is bounded,

corollary 2.2.11 implies $|S| \leqslant 2^{\aleph_0}$ and so $a \in bdd(A)$. Finally, lemma 2.2.10 gives us $a \in acl_{\mathcal{M}}(B)$.

(3) Let C be the compact subset witnessing $a \in acl_{\mathcal{M}}(A)$, the definability of C only depends in a countable set of $L(A)$-functions, since the length of each formula is finite, it only depends in a countable $A_0 \subseteq A$. Hence, $a \in acl_{\mathcal{M}}(A_0)$.

(4) Statement (2) implies $acl_{\mathcal{M}}(A) \subseteq acl_{\mathcal{M}}(B)$. Now, we are going to prove $acl_{\mathcal{M}}(B) \subseteq acl_{\mathcal{M}}(A)$. Let $a \in acl_{\mathcal{M}}(B)$ and let $C \subseteq M$ be a compact set definable in $\mathcal{M}$ over B containing a. Hence, the definable predicate $\mathrm{dist}(x, C)$ is the limit of the interpretations of a sequence of $L(B)$-formulas $(\varphi_n : n \geqslant 1)$. Let a_n be the tuple of elements of B that occur in φ_n. Since A is dense in B, for every n there exists a sequence of tuples of elements in A $(a_n^{(k)} : k \geqslant 1)$ converging to a_n. Taking a subsequence if necessary, we may assume that for every $n \geqslant 1$, $|\varphi_n^{\mathcal{M}}(x, a_n) - \mathrm{dist}(x, C)| < \frac{1}{2n}$ for all tuples x of elements in M. Let Δ_n be the modulus of uniform continuity of $\varphi_n^{\mathcal{M}}$. Taking a subsequence if necessary, we may assume that for any each $n \geqslant 1$, $d(a_n, a_n^{(k)}) < \Delta_n(\frac{1}{2n})$ for $k \geqslant n$. This implies that $|\varphi_n^{\mathcal{M}}(x, a_n) - \varphi_n^{\mathcal{M}}(x, a_n^{(n)})| < \frac{1}{2n}$ for all tuples x of elements in M. Hence, applying the triangular inequality,

$$|\mathrm{dist}(x, C) - \varphi_n^{\mathcal{M}}(x, a_n^{(n)})| \leqslant |\varphi_n^{\mathcal{M}}(x, a_n) - \mathrm{dist}(x, C)| + |\varphi_n^{\mathcal{M}}(x, a_n) - \varphi_n^{\mathcal{M}}(x, a_n^{(n)})| \leqslant \frac{1}{n}$$

for every tuple x of elements in M. This imply that the interpretations of the sequence of $L(A)$-formulas $(\varphi_n(x, a_n^{(n)}) : n \geqslant 1)$ converge uniformly to $\mathrm{dist}(x, C)$ on $\mathcal{M}$ and hence C is definable in $\mathcal{M}$ over A. □

Proposition 2.2.13. *Let $\mathcal{M}$ be an L-structures with $A, B \subseteq M$. Then, any elementary map $\alpha : A \to B$ extends to an elementary map α' from $acl_{\mathcal{M}}(A)$ into $acl_{\mathcal{M}}(B)$. Moreover, if α is surjective, then so is α'*

Proof. Let $\mathcal{M}'$ be an elementary extension of $\mathcal{M}$ sufficiently saturated and strongly homogeneous. The homogeneity of $\mathcal{M}'$ implies that α extends to an automorphism g of $\mathcal{M}'$. Since automorphisms of metric structures are continuous functions (they are Lipschitz with constant 1), if $C \subseteq M$ is a compact set, so is $g(C) \subseteq M$. Moreover, if $(\varphi_n(x, a_n^1, \ldots, a_n^{m(n)}) : n \geqslant 1)$ is the sequence of $L(A)$-formulas witnessing the definability of $\mathrm{dist}(x, C)$ on M, then the sequence of $L(B)$-formulas $(\varphi(x, g(a_n^1), \ldots, g(a_n^{m(n)})) : n \geqslant 1)$ witness that $g(C)$ is definable in $\mathcal{M}$ over A. Hence, $g(acl_{\mathcal{M}}(A)) \subseteq alc_{\mathcal{M}}(B)$. Thus, taking the restriction of g to $acl_{\mathcal{M}}(A)$ we get the required elementary map α'.

If $\alpha(A) = B$, then for any compact set C definable in $\mathcal{M}$ over B, the definable predicate can be written as the limit of the interpretations of a sequence of formulas $(\psi_n(x, f(a_n)) : n \geqslant 1)$, where the a_n are tuples of elements of A not necessarily all of the same length. This implies that $C = g(K)$ for some set K definable in $\mathcal{M}$ over A. Since g^{-1} is also an automorphism of $\mathcal{M}'$, g^{-1} is a continuous function and hence K is compact. Therefore, $g(acl_{\mathcal{M}}(A)) = acl_{\mathcal{M}}(B)$.

□

CHAPTER 3
Further work

In this chapter, we study applications of model theory for metric structures to Hilbert spaces over $\mathbb{R}$, that is, real vector spaces equipped with an inner product and complete with respect to the corresponding norm. In continuous logic, we identify them with many sorted metric structures structures. That is, we identify a HIlbert space H with

$$\mathcal{M}(H) = ((B_n(H) : n \geqslant 1), 0, \{I_{n,m}\}_{n<m}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\cdot\rangle),$$

where $B_n(H) = \{x \in H : \langle x, x\rangle \leqslant n^2\}$ for all $n \geqslant 1$, each sort with the metric induced by the norm; 0 is the zero vector in $B_1(H)$; for $n < m$ $I_{n,m} : B_m \to B_n$ is the inclusion map; for $r \in \mathbb{R}$, $n \geqslant 1$ and and the unique $k \geqslant 1$ satisfying $k - 1 \leqslant |r| < k$, $\lambda_r : B_n \to B_{nk}$ is the scalar multiplication by r; furthermore, $\langle\cdot\rangle : B_n(H) \to [-n^2, n^2]$ is the inner product for each $n \geqslant 1$ and the functions $+ : B_n(H) \times B_n(H) \to B_{2n}(H)$ and $- : B_n(H) \times B_n(H) \to B_{2n}(H)$ are vector addition and subtraction respectively.

It is easy to construct a signature L for which each many sorted structure $\mathcal{M}(H)$ as above is an L-structure since the bounds on the metric spaces are trivial and the moduli Δ of uniform continuity are easy to specify.

By proposition 1.2.14 it can be proved that the class of Hilbert spaces is axiomatizable (see [5, page 90])

Let IHS be the L-theory obtained by adding to the theory of Hilbert spaces the L-conditions

$$\inf_{x_1} \ldots \inf_{x_n} \max_{1\leqslant i,j\leqslant n} (|\langle x_i, x_j\rangle - \delta_{i,j}|) = 0$$

for all $n \geqslant 1$, where the variables range over the sort $B_1(H)$. Then it is clear that any model of IHS is isomorphic to $\mathcal{M}(H)$ for some infinite dimensional Hilbert space H.

If A is a subset of a Hilbert space H, we denote $\overline{A}$ to the norm closure of the linear span of A and $A^\perp$ to the orthogonal complement of A. A well known property of Hilbert spaces is the orthogonal decomposition $H = \overline{A} \oplus A^\perp$. We denote $P_{\overline{A}}(x)$ the projection on the subspace $\overline{A}$ of the element x.

We are going to prove that $\overline{A}$ coincides with the definable closure of A, to do so, we need first a lemma.

Lemma 3.0.1. *Let H be an infinite dimensional Hilbert space, and let $c_1, \dots, c_n$, $d_1, \dots, d_n \in H$. Then, $(c_1, \dots, d_1)$ and $(d_1, \dots, d_n)$ realize the same type over $A \subseteq H$ if and only if $P_{\overline{A}}(c_i) = P_{\overline{A}}(d_i) =$ and $\langle c_i, c_j \rangle = \langle d_i, d_j \rangle$ for al $1 \leqslant i, j \leqslant n$.*

Proof. Assume that $\mathrm{tp}(c_1, \dots, c_n/A) = \mathrm{tp}(d_1, \dots, d_n/A)$. Then, $\langle c_i, c_j \rangle = \langle d_i, d_j \rangle$ for $1 \leqslant i, j \leqslant n$. Furthermore, for every $a, b \in A$ $\langle c_i - b, a \rangle = \langle d_i - b, a \rangle$ since $\langle x_i - b, a \rangle = r$ belongs to $\mathrm{tp}(c_1, \dots, c_n/A)$ for some $r \in [0, 1]$. This implies $P_{\overline{A}}(c_i) = P_{\overline{A}}(d_i)$ for all $i = 1, \dots, n$.

Assume now that $P_{\overline{A}}(c_i) = P_{\overline{A}}(d_i)$ and $\langle c_i, c_j \rangle = \langle d_i, d_j \rangle$ for all $1 \leqslant i, j \leqslant n$. Then we have that $c_i - P_{\overline{A}}(c_i)$ and $d_i - P_{\overline{A}}(d_i)$ belongs to $A^\perp$, hence $d_i - c_i \in A^\perp$ for all $i = 1, \dots, n$. Furthermore,

$$\langle c_i - P_{\overline{A}}(c_i), c_j - P_{\overline{A}}(c_j) \rangle = \langle d_i - P_{\overline{A}}(d_i), d_j - P_{\overline{A}}(d_j) \rangle,$$

for any $1 \leqslant i, j \leqslant n$. This means that we can construct an $\overline{A}$-isomorphism between the subspace generated by $c_1, \dots, c_n$ and the subspace generated by $d_1, \dots, d_n$ sending c_i to d_i for $i = 1, \dots, n$. Using Gram-Schmidt (which in arbitrary cardinalities is obtained using Zorn's Lemma) we can extend this isomorphism between the respective subspaces to an $\overline{A}$-automorphism of H taking c_i to d_i for all $i = 1, \dots, n$. □

Proposition 3.0.2. *Let H be an infinite dimensional Hilbert Space and let $A \subseteq H$. Then, $dcl(A) = \overline{A}$.*

Proof. We may assume that $\overline{A}$ is a proper subspace of H, passing to an elementary extension if needed. Note that by lemma 2.2.3, passing to an elementary extension does not change $dcl(A)$.

Assume that $c \in \overline{A}$. Then, there exists a Cauchy sequence $(c_n : n \geqslant 1)$ of elements in $span(A)$ such that $\lim_{n \to \infty} c_n = c$. Taking a subsequence if necessary we may assume $\|c - c_n\| \leqslant \frac{1}{2n}$ for all $n \geqslant 1$. Finally, by proposition 2.1.12 or proposition 2.2.6, the family of formulas $\varphi_n := \|x - c_n\| \dot{-} \frac{1}{2n}$ and numbers $\{\delta_n = \frac{1}{2n}\}$ witness that $\{c\}$ is a definable set over A.

Assume now that $c \notin \overline{A}$. Then, $c - P_{\overline{A}}(c) \neq 0$. Let $y \in A^\perp$ be such that $\|y\| = \|c - P_{\overline{A}}(c)\|$. Then, by lemma 3.0.1, $\mathrm{tp}(c/A) = \mathrm{tp}((P_{\overline{A}}(c) + y)/A)$. Since $A^\perp$ is nonempty by assumption, this shows that there exists at least one realization of $\mathrm{tp}(c/A)$ in H that is different from c and hence by proposition 2.2.6, $c \notin dlc(A)$. □

We give a result relating explicitly the distance in the type space with the norm of the realizations.

Proposition 3.0.3. *Let H be an infinite dimensional Hilbert Space. For each $x, y \in H$ and $A \subseteq H$ the following equality holds:*

$$d(\mathrm{tp}(x/A), \mathrm{tp}(y/A))^2 = \|P_{\overline{A}}(x) - P_{\overline{A}}(y)\|^2 + |\|x - P_{\overline{A}}(x)\| - \|y - P_{\overline{A}}(y)\||^2.$$

Proof. Let $x, y, x', y' \in H$ and let $A \subseteq H$ be such that $\mathrm{tp}(x/A) = \mathrm{tp}(x'/A)$ and $\mathrm{tp}(y/A) = \mathrm{tp}(y'/A)$. Then,

$$\begin{aligned}\|x' - y'\| = \|P_{\overline{A}}(x') - P_{\overline{A}}(y')\|^2 + \|(x' - P_{\overline{A}}(x')) - (y' - P_{\overline{A}}(y'))\| \geqslant \\ \|P_{\overline{A}}(x') - P_{\overline{A}}(y')\|^2 + \|\|x' - P_{\overline{A}}(x')\| - \|y' - P_{\overline{A}}(y')\|\|^2 = \\ \|P_{\overline{A}}(x) - P_{\overline{A}}(y)\|^2 + \|\|x - P_{\overline{A}}(x)\| - \|y - P_{\overline{A}}(y)\|\|^2,\end{aligned}$$

where the last equality follows from 3.0.1. Since $d(\mathrm{tp}(x/A), \mathrm{tp}(y/A))$ is realized at some pair of elements, the inequality above implies

$$d(\mathrm{tp}(x/A), \mathrm{tp}(y/A))^2 \geqslant \|P_{\overline{A}}(x) - P_{\overline{A}}(y)\|^2 + \|\|x - P_{\overline{A}}(x)\| - \|y - P_{\overline{A}}(y)\|\|^2.$$

For the other inequality, we denote $x_\perp = x - P_{\overline{A}}(x)$ and $y_\perp = y - P_{\overline{A}}(y)$. We may assume $x_\perp \neq 0$ since for $x_\perp = 0$ the result is trivial. Let $\alpha = \frac{\|y_\perp\|}{\|x_\perp\|}$ and let $z = \alpha x_\perp$. From lemma 3.0.1 it follows that $\mathrm{tp}(y/A) = \mathrm{tp}((P_{\overline{A}}(y) + z)/A)$. Hence,

$$\begin{aligned}&d(\mathrm{tp}(x/A), \mathrm{tp}(y/A))^2 \leqslant \|x - (P_{\overline{A}}(y) + z)\|^2 = \\ &\|P_{\overline{A}}(x) - P_{\overline{A}}(y)\|^2 + \|z_\perp - \alpha x_\perp\|^2 = \|P_{\overline{A}}(x) - P_{\overline{A}}(y)\|^2 + \|\|x_\perp\| - \|y_\perp\|\|^2\end{aligned}$$

□

We also introduce some results that are out of the scope of this memoir.

Proposition 3.0.4. [5, page 90] *IHS is a complete theory.*

We say that an L-formula $\varphi(x_1, \ldots, x_n)$ is *approximable in* T by *quantifier-free formulas* if for every $\varepsilon > 0$ there is a quantifier-free L-formula $\psi(x_1, \ldots, x_n)$ such that for all $\mathcal{M} \models T$ and all $a_1, \ldots, a_n \in M$, we have

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leqslant \varepsilon.$$

We say that an L-theory T admits *quantifier elimination* if every L-formula is approximable in T by quantifier-free formulas.

Corollary 3.0.5. [5, corollary 15.2] *The theory IHS admits quantifier elimination.*

We say that an L-theory T is λ-*stable* if for any $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| \leqslant \lambda$, there exists a dense subset in $S_1(T_A)$ (with respect to the d-metric) of cardinality less or equal than λ.

Proposition 3.0.6. [5, proposition 15.5] *The theory IHS is* ω*-stable.*

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