The Ševera-Roytenberg Correspondence

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Abstract

This work is an introduction to the theory of graded manifolds with particular emphasis on its relation with generalized geometry and the study of Courant algebroids. First we present graded manifolds and extend many constructions from ordinary differential geometry to this setting. Then we provide an overview of the main examples of graded manifolds appearing in the literature. We prove Vaintrob's Theorem characterizing Lie algebroids as NQ-manifolds of degree 1 and Ševera-Roytenberg's Theorem characterizing Poisson manifolds as symplectic NQ-manifolds of degree 1 and Courant algebroids as symplectic NQ-manifolds of degree 2. We also show that the deformation theory of a Courant algebroid is naturally described by the Q-cohomology of its corresponding Q-manifold. Finally, a new construction of a graded Poisson NQ-manifold associated to a Courant algebroid is presented. As an application, we obtain a Bianchi identity for the curvature of a generalized connection.

Contents

1 Introduction **I**

1.1 Courant Algebroids and σ-Models I

1.2 Graded Geometry IV

1.3 Outline of this Work VIII

2 Graded Manifolds **1**

2.1 Graded Algebra 1

2.2 Graded Manifolds 6

2.3 Vector Fields and Differential Forms 12

2.4 Morphisms of Graded Manifolds 15

3 Classes of Graded Manifolds **19**

3.1 Derived Brackets 19

3.2 L_∞-algebras 22

3.3 Q-manifolds 26

3.4 Graded Symplectic Manifolds 30

3.5 N-manifolds and Lie Algebroids 35

4 The Ševera-Roytenberg correspondence **41**

4.1 Courant Algebroids 41

4.2 Symplectic N-manifolds of Degree 1 and 2 48

4.3 Symplectic NQ-manifolds of Degree 1 and 2 54

4.4 Courant Algebroids & Graded Geometry 57

4.5 Generalized Riemannian Geometry 63

4.6 Generalized Riemannian Geometry & Graded Geometry 68

CHAPTER 1

Introduction

1.1. Courant Algebroids and σ-Models

The main motivation for the study of Courant algebroids comes from two-dimensional variational problems. Of particular importance are those that appear in physics under the language of σ-models for string field theories. We devote this section to make a brief introduction to the terminology of this area because it will give us intuition and motivation through the whole work.

A general problem in physics is to predict the value of a **field** in terms of the forces that is is subject to. A field here is simply an assignment of a physical magnitude to each point of some space. The Lagrangian approach to this problem consists on considering the set F of all possible fields and defining an **action functional** $S : F \to \mathbb{R}$ representing some kind of energy involved in the actual realization of each particular field. The **least action principle** asserts that the fields that are physically realized are the ones that, at least locally, minimize S; hence, the problem is reduced to obtaining the critical points of S.

In classical mechanics the problem of interest is to predict the movement of a particle, modelled as a point, subject to some forces in spacetime. The resulting field theories are called **σ-models**. Here $F = Map(I, M)$ for $I = [a, b] \subset \mathbb{R}$ a time interval — the **source** or **worldline** — and M a manifold representing spacetime — the **target** — and the action functional $S : Map(I, M) \to \mathbb{R}$ usually takes the form

$$S(\varphi) = \int_I \mathcal{L}(t, \varphi, \varphi', ..., \varphi^{(r)})\, dt, \qquad \varphi \in Map(I, M),$$

where the **Lagrangian density** $\mathcal{L}$ is a function of t, φ and its derivatives up to order r; that is, $\mathcal{L} : I \times J^r M \to \mathbb{R}$, where $J^r M$ is the rth jet bundle of M. For example, a choice of Riemannian metric g on M can be used to define

$$\mathcal{L}(\varphi, \varphi') = \frac{1}{2}|\varphi'|_g^2 + V(\varphi)$$

for $V : M \to \mathbb{R}$ a potential; in this case we can think of $S(\varphi)$ as the total energy (kinetic plus potential) consumed along the trajectory φ. Performing an integration by parts we see that the critical points of a general S are given, at least formally, by the **Euler-Lagrange equation**

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \varphi'} + \frac{\partial^2}{\partial t^2}\frac{\partial \mathcal{L}}{\partial \varphi''} - \ldots + (-1)^r \frac{\partial^r}{\partial t^r}\frac{\partial \mathcal{L}}{\partial \varphi^{(r)}} = 0.$$

If $\varphi_0 \in Map(I, M)$ satisfies this equation, then it is still not true that φ minimizes S locally. What is true is that the variation of S along a family $\{\varphi_\epsilon\}_\epsilon$ starting at φ_0 depends exclusively on the variation of φ_ϵ at ∂I; hence, imposing appropriate boundary conditions on the solutions this variation is zero and we can conclude that solutions to the Euler-Lagrange equation give critical points of S.

We can rephrase this in a more geometric language as follows (see [25] for a very complete exposition of calculus of variations in terms of differential forms). The general form of an action functional $S : Map(I, M) \to \mathbb{R}$ is

$$S(\varphi) = \int_I \varphi^* \tau$$

for $\tau = \mathcal{L}(t, q, v_1, \ldots, v_r)dt$ a one-form on $N = I \times J^r M$, where we are making a small abuse of notation when writing φ for the map $t \mapsto (t, \varphi(t), \varphi'(t), \ldots, \varphi^{(r)}(t))$. Let us consider then the abstract problem of minimizing $\int_I \varphi^* \tau$ over maps $\varphi : I \to N$ for a fixed one-form $\tau \in \Omega^1(N)$. We can identify $T_\varphi Map(I, N) \cong \Gamma(\varphi^* TN)$ and compute the variation of S at φ along the direction $\varphi^* X \in \Gamma(\varphi^* TN)$ as

$$\delta S(\varphi^* X) = L_{\varphi^* X}(S)(\varphi) = \int_I \varphi^*(L_X \tau) = [\varphi^* \iota_X \tau]_{t=a}^{t=b} + \int_I \varphi^* \iota_X d\tau,$$

where L_X stands for the Lie derivative. Fixing boundary conditions amounts precisely to considering only $X \in \Gamma(TN)$ such that $\varphi^* \iota_X \tau = 0$ at ∂I; hence, the Euler-Lagrange equation is $\varphi^* \iota_X d\tau = 0$ for all such $X \in \Gamma(TN)$. Assume that φ is such that, in fact, $\varphi^* \iota_X d\tau = 0$ for all $X \in \Gamma(TN)$ and suppose that we have a pair $(X, f) \in \Gamma(TN) \times C^\infty(N)$ satisfying $L_X \tau - df = 0$. Then it follows from Cartan's formula for L_X that $d\varphi^*(\iota_X \tau - f) = 0$; that is, the quantity $\iota_X \tau - f$ is conserved along these solutions of the variational problem.

Noether's Theorem is a general principle in physics that states that conserved quantities are usually a consequence of an invariance of the Lagrangian density $\mathcal{L}$ under a group action on the space of possible Lagrangian densities (here, this is $\Omega^1(N)$), so it would be interesting to describe this action in a geometric way. By this we mean that we want to describe an action of $(X, f) \in \Gamma(TN) \times C^\infty(N)$ — written in what follows as $X + f \in \Gamma(TN \oplus \mathbb{R})$ — on $\Omega^1(M)$ leaving invariant those forms $\tau \in \Omega^1(M)$ such that $L_X \tau - df = 0$. One way to do this is to represent each $\tau \in \Omega^1(M)$ as the subbundle $D_\tau := \{Y + \iota_Y \tau \in \Gamma(TN \oplus \mathbb{R}) : Y \in \Gamma(TN)\}$ and define the following bracket on $\Gamma(TN \oplus \mathbb{R})$:

$$[X + f, Y + g] := [X, Y] + (X(g) - Y(f)).$$

It is easy to check that $[\cdot, \cdot]$ is a skew-symmetric bracket such that $[X + f, \cdot]$ leaves D_τ invariant if and only if $L_X \tau - df = 0$. This bracket is quite natural: X acts via L_X on Y and g and Y acts via $-L_Y$ on X and f. One can easily check that $[\cdot, \cdot]$ satisfies the Jacobi identity and the same Leibniz rule as the Lie bracket of vector fields, showing that $\Gamma(TN \oplus \mathbb{R})$ is a **Lie algebroid** — we will define these in Section 3.5. As we have seen, a good geometric understanding of this structure can help in the study of solutions to variational problems.

Now in string theory particles are no longer modelled as points, but as **strings**. This means that trajectories are described in this setting by elements of $Map(\Sigma, M)$ for Σ — the **worldsheet** — a smooth compact oriented surface with a Riemannian metric η representing the string and its internal time. The usual form of an action functional is now $S(\varphi) = \int_\Sigma \varphi^*\tau$ for τ a 2-form on $N = \Sigma \times J^r M$, and the same computations as before show that, for a pair $X + \alpha \in \Gamma(TN \oplus T^*N)$ such that $L_X\tau - d\alpha = 0$, the one-form $\iota_X\tau - \alpha$ is conserved (i.e., closed) along solutions of the Euler-Lagrange equation.

Courant algebroids appear when studying the pairs $X + \alpha \in \Gamma(TN \oplus T^*M)$ from a geometric point of view analogous to what we have done above for $\Gamma(TN \oplus \mathbb{R})$. If we associate to $\tau \in \Omega^2(M)$ the subbundle $D_\tau = \{Y + \iota_Y\tau \in \Gamma(TN \oplus T^*N) \,:\, Y \in \Gamma(TN)\}$ and define

$$[X + \alpha, Y + \beta] := [X, Y] + L_X\beta - \iota_Y d\alpha,$$

then $[X + \alpha, \cdot]$ preserves D_τ if and only if $L_X\tau - d\alpha = 0$. After performing some computations one can show that this (non-skew) bracket has many interesting properties: it satisfies the Jacobi identity and a Leibniz rule and it is equivariant with respect to the canonical pairing on $TN \oplus T^*N$. A good geometric understanding of this structure helps understand the properties of the solutions to the Euler-Lagrange equations of the given problem and the global structure of the whole space of such solutions. For example, notice that the Euler-Lagrange equation $\varphi^*\iota_X d\tau = 0$ does not change if we substitute τ by $\tau + \beta$, for β a closed 2-form. This is reflected in $TN \oplus T^*N$ by the fact that $X + \alpha \mapsto X + \alpha + \iota_X\beta$ is an automorphism of $TN \oplus T^*N$ preserving the bracket $[\cdot, \cdot]$, the pairing $\langle \cdot, \cdot \rangle$ and the projection $X + \alpha \to X$. Of course, one can also go on and define similar brackets on each $TN \oplus \Lambda^k T^*N$, which will describe the structure of $(k + 1)$-dimensional variational problems.

A vector bundle $E \to N$ with a non-degenerate pairing $\langle \cdot, \cdot \rangle$, an anchor $a : E \to TN$ (for $E = TN \oplus T^*N$ this is the projection $X + \alpha \to X$) and a bracket $[\cdot, \cdot] : E \times E \to E$ satisfying the same properties as those in $TN \oplus T^*N$ is called a **Courant algebroid**. Asides from $TN \oplus T^*N$ with the above structure, other more exotic Courant algebroids appear naturally in two-dimensional σ-models in which some symmetries or twists are to be taken into account.

Most of classical differential geometry is concerned with the study of a manifold N by means of some additional structure defined on its tangent bundle TN (say, a pseudo-Riemannian metric, a symplectic form, a complex structure, etc.), and the Lie bracket of vector fields on TN usually plays an important role on describing some notion of integrability of these structures. In **generalized geometry** TN is substituted by $TN \oplus T^*N$ with the bracket $[\cdot, \cdot]$ or, more generally, by any Courant algebroid over N. This allows to characterize the geometry of N in terms of new structures: **generalized metrics**, **generalized complex structures**, **Dirac structures**, etc. For example, generalized complex structures interpolate symplectic and complex structures on N and Dirac structures on $TN \oplus T^*N$ interpolate presymplectic and Poisson structures on N.

The bracket $[\cdot,\cdot]$ on $TN \oplus T^*N$ and its skew-symmetrization appeared first in [14] and [11] as a tool for studying from a unifying point of view the equations $d\alpha = 0$ for $\alpha \in \Gamma(\Lambda^2 T^*N)$ (responsible for presymplectic structures on M) and $[\pi,\pi] = 0$ for $\pi \in \Gamma(\Lambda^2 TN)$ (responsible for Poisson structures on M), where $[\cdot,\cdot]$ is the Schouten bracket. It was later noticed in [37], where Courant algebroids were baptised, that the natural analog for Lie bialgebroids of the **Drinfeld double** of a Lie bialgebra is a Courant algebroid. Then Ševera became interested in them [50] as a model for **Poisson-Lie T-duality** and pointed out their relation with two-dimensional variational problems and string theory. Generalized complex geometry was introduced by Hitchin in [26] while studying special geometry in low dimension and is now a very active field of research; a good survey on the basics of this area can be found in [22].

1.2. Graded Geometry

Let us briefly describe what a graded manifold is. One way to define ordinary differentiable manifolds is through their sheaf of functions: namely, a C^∞ manifold $(M, C^\infty(M))$ is a Hausdorff, second countable topological space M with a sheaf $C^\infty(M)$ of commutative algebras whose localization at every $p \in M$ is a local ring and such that there exists an open cover $\{U_\alpha\}_\alpha$ of M and isomorphisms of locally ringed spaces $\varphi_\alpha : (U_\alpha, C^\infty(M)_{|U_\alpha}) \to (V_\alpha, C^\infty(V_\alpha))$ for some open sets $V_\alpha \subset \mathbb{R}^n$. A **graded manifold** $\mathcal{M} = (M, C^\infty(\mathcal{M}))$ can be defined in the same way, but now $C^\infty(\mathcal{M})$ is a sheaf over M of graded (say, in $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$) supercommutative algebras, the local model is $C^\infty(V_\alpha) \otimes \mathbb{R}[\xi^1, ..., \xi^d]$ for ξ^1, ..., ξ^d supercommuting variables of non-zero degrees, and the maps φ_α are required to preserve the degrees. If $p(f) \in \mathbb{Z}/2\mathbb{Z}$ denotes the **parity** of $f \in C^\infty(\mathcal{M})$, by **supercommutativity** we mean that $fg = (-1)^{p(f)p(g)} gf$. With this language, one can also define in a natural way vector fields, differential forms, vector bundles, connections, etc. for graded manifolds.

The simplest examples of graded manifolds are vector bundles: if $E \to M$ is a vector bundle of rank r, the transition maps of E are maps $C^\infty(U_\alpha)\otimes\mathbb{R}^r \to C^\infty(U_\beta)\otimes\mathbb{R}^r$ which can be extended to isomorphisms of graded algebras $C^\infty(U_\alpha)\otimes\mathbb{R}[\xi^1, ..., \xi^r] \to C^\infty(U_\beta)\otimes\mathbb{R}[\xi^1, ..., \xi^r]$, where the coordinates ξ^i are all assigned a fixed degree $(k,\epsilon) \in \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \setminus \{(0,0)\}$, and these isomorphisms can be seen as the coordinate changes of a graded manifold $\mathcal{M}$ over M. If $\epsilon = 1$, then $C^\infty(\mathcal{M}) = \Gamma(\Lambda^* E^*)$; if $\epsilon = 0$, then $C^\infty(\mathcal{M}) = \Gamma(S^* E^*)$. This graded manifold is quite simple because the graded coordinates ξ^i transform linearly between themselves. In general graded manifolds, other transformations are allowed, such as $x_\alpha \mapsto x_\beta + \xi_\beta\eta_\beta$ for variables of degrees $deg(x_\alpha) = deg(x_\beta) = (0,0)$ and $deg(\xi_\beta) = -deg(\eta_\beta) \in \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \setminus \{(0,0)\}$.

The idea of grading in algebra has been around for a long time now; originally it was just a way of organizing the information either for performing inductive arguments or for establishing sign rules for commutativity relations. First appearances of a $\mathbb{Z}$-grading in geometric structures can be traced back to the ***BRST* formalism**, an attempt of quantizing field theories with symmetries which involved the introduction of **ghost fields** not representing any physical magnitude as a technical requirement for the theory to work. In the modern language of graded geometry, ghost fields are simply *functions of non-zero degree* on a graded manifold.

On the other hand, $\mathbb{Z}/2\mathbb{Z}$-grading appeared in physics as a way of distinguishing between **bosons** and **fermions**. Bosons are particles with integer spin, while fermions are particles with half-integer spin, and the spin of the composite of two particles is the sum of the original spins. Thus, composing an even number of fermions gives a boson but composing any number of bosons gives a boson. This is naturally modelled as a $\mathbb{Z}/2\mathbb{Z}$-grading, where bosons have parity 0 and fermions have parity 1. This $\mathbb{Z}/2\mathbb{Z}$-grading is particularly important when modelling **supersymmetry**. Field theories of interest in physics are usually required to be invariant in some way or another by the Poincaré group, which is the group of transformations preserving the Minkowski metric of spacetime; i.e., the semidirect product of $SO(3,1)$ and the group of $\mathbb{R}^4$-translations. It was then noticed that many of the two-dimensional σ-models appearing in string theory were in fact invariant by the **super**-Poincaré group, an extension of the Poincaré group including a new type of symmetry based on the interaction between bosons and fermions, and it was conjectured that supersymmetry was a general phenomenon in physics.

In any case, graded manifolds seemed to appear naturally in field theories. In the origins of this theory the difference between $\mathbb{Z}$-grading and $\mathbb{Z}/2\mathbb{Z}$-grading was not very clear, but now the literature usually distinguishes between **super**-structures — those with a $\mathbb{Z}/2\mathbb{Z}$-grading — and **graded** structures — those with a $\mathbb{Z}$-grading —, each form of grading playing a different role. Of particular importance is Berezin's program on defining the $\mathbb{Z}/2\mathbb{Z}$-graded versions of every object in mathematics, including the theory of integration on super-vector spaces which is later extended to supermanifolds [36]. In recent years the theory of graded manifolds has attracted attention because it models complicated structures in a handy geometric language revealing new facets of these objects. Let us discuss two examples of this idea.

As we have seen in Section 1.1, the Lagrangian approach to classical mechanics consists on describing the trajectories of a particle moving in spacetime M as solutions $\varphi \in Map(I, M)$ to a differential equation described by a lagrangian density $\mathcal{L} : I \times J^r M \to \mathbb{R}$. In many situations, for each fixed time $t_0 \in I$, $\mathcal{L}$ is only a function of TM (for example, $\mathcal{L} = \frac{1}{2}|\varphi'|_g^2 + V(\varphi)$) and we can then say that the **phase space** of the system is TM, meaning that the state of the physical system at each $t_0 \in I$ is completely described by an element of TM. The **Legendre transform** induced by $\mathcal{L} \in C^\infty(TM)$ is the isomorphism

$$\begin{aligned} F\mathcal{L} : TM &\to T^*M \\ (p, v) &\mapsto \left(p, \frac{\partial}{\partial v}\mathcal{L}(p, \cdot)\right) \end{aligned}$$

which allows us to see T^*M as the phase space; in the case of $\mathcal{L} = \frac{1}{2}|\varphi'|_g^2 + V(\varphi)$ this is just the identification of TM and T^*M through the metric g. The appearance of symplectic manifolds as models for the phase space is a general and very useful feature of classical mechanics. These are not always cotangent bundles, but usually they do appear naturally as reductions of these. For example, if the target manifold is $\mathbb{R}^4$ with the Minkowski metric, many field theories are required to be $SO(3)$-invariant; this means that the angular momentum $\mu : T^*\mathbb{R}^4 \to \mathfrak{so}(3)$ is preserved and so for a fixed value $v \in \mathfrak{so}(3)$ the mechanics are really happening on the symplectic reduction $\mu^{-1}(v)//SO(3) \cong S^2$ rather than on $T^*\mathbb{R}^4$. In some cases, however, a symplectic reduction is not possible and similar reduction proceedings give **Poisson** manifolds instead, which satisfy less useful properties.

Let us move to the context of string theory. Now trajectories are described by $\varphi \in Map(\Sigma, M)$ for Σ a Riemann surface describing a string S^1 moving in time. If the Lagrangian $\mathcal{L}$: $\Sigma \times J^r M$ only depends at each fixed time on TM, we can say that the phase space of the system is $Map(S^1, TM)$ and we can use again the Legendre transform to identify this with $Map(S^1, T^*M)$. This is an infinite-dimensional manifold but it is still symplectic: the tangent space of $Map(S^1, T^*M)$ at φ can be identified with $\Gamma(\varphi^*T(T^*M))$ and so we can define the symplectic form Ω at $T_\varphi Map(S^1, T^*M)$ as

$$\Omega(\varphi^*X, \varphi^*Y) := \int_{S^1} \omega(X(\varphi(\theta)), Y(\varphi(\theta)))\, d\theta,$$

for $d\theta$ a measure on S^1 and ω the symplectic form on T^*M. An interesting remark is that sections of $TM \oplus T^*M$ can be naturally identified with functions over $Map(S^1, T^*M)$. Indeed, for $\varphi \in Map(S^1, T^*M)$ and $X + \alpha \in \Gamma(TM \oplus T^*M)$ we can define

$$(X + \alpha)(\varphi) := \int_{S^1} \varphi(\theta)(X) d\theta + \int_{S^1} \varphi^*\alpha\,.$$

As seen in Section 1.1, $TM \oplus T^*M$ is a Courant algebroid, and this structure is strongly related to two-dimensional variational problems. Hence, if the infinite-dimensionality of $Map(S^1, T^*M)$ constitutes a problem for its study as a phase space, it makes sense to turn our attention to $TM \oplus T^*M$ at least as a toy model. A good understanding of the geometry of $TM \oplus T^*M$ that takes its Courant algebroid structure into account can give us first ideas on how to study $Map(S^1, T^*M)$.

The theorem that gives name to this work is Rotenberg's result [42] on the characterization of Poisson manifolds and Courant algebroids as **symplectic NQ-manifolds** of degree 1 and 2, respectively. A symplectic NQ-manifold of degree k is a triple $(\mathcal{M}, \omega, \Theta)$ where $\mathcal{M}$ is a non-negatively graded manifold, ω is a symplectic form of degree k on $\mathcal{M}$ and Θ is a function of degree $k+1$ satisfying $\{\Theta, \Theta\} = 0$ under the Poisson bracket of $\mathcal{M}$. This suggests a direction in which methods for performing reduction or quantization of Poisson manifolds and Courant algebroids can be attempted: by extending the ideas of symplectic reduction and geometric quantization from ordinary symplectic manifolds to graded symplectic manifolds. There is indeed some work done from this perspective, related to multisymplectic geometry and higher Chern-Simmons theory: see for example [8], [40], [16] .

A method for performing a (graded analog of a) deformation quantization of the symplectic NQ-manifold $(\mathcal{M}, \omega, \Theta)$ associated to a Courant algebroid E is presented in [21]. Interestingly, this method reveals some links [19], [52] with constructions arising from the generalized Riemannian geometry of E: the quantization of the function $\Theta \in C^\infty(\mathcal{M})$ is an operator on a spinor bundle of E, called the **canonical Dirac operator**, and which can be defined in terms of torsion-free generalized connections on E. It would be interesting to study the interplay of this relation with generalized metrics and, in particular, the appearance of the (generalized) Riemannian, Ricci and scalar curvature tensors of E on this method for quantizing $\mathcal{M}$.

The language of graded geometry also allows to define **AKSZ σ-models** (see [27], [9] for detailed expositions of this subject and [48], [43] for some original articles, here we will just give a brief description of the philosophy of this formalism). The AKSZ σ-model with target an NQ-manifold $(\mathcal{M}, \omega, \Theta)$ of degree k is a field theory for mechanics with source a $(k+1)$-dimensional ordinary manifold Σ. The space of fields of this theory is the set of morphisms of graded manifolds $Mor(T[1]\Sigma, \mathcal{M})$, where $T[1]\Sigma$ is the shifted tangent bundle of Σ (functions on $T[1]\Sigma$ are differential forms on Σ). If $\{x^a\}_a$ are local coordinates on $\mathcal{M}$ such that $\omega = \frac{1}{2}\omega_{ab} dx^a \wedge dx^b$ for $\omega_{ab} \in \mathbb{R}$, a field $\varphi \in Mor(T[1]\Sigma, \mathcal{M})$ is described by the differential forms $\varphi^* x^a \in \Omega(\Sigma)$, which satisfy $\varphi^* x^a \in \Omega^p(\Sigma)$ whenever $deg(x^a) = p$. The action functional of the theory can be described in these coordinates as $S = S_0 + S_1$ with

$$S_0(\varphi) = \int_\Sigma \frac{1}{2}\omega_{ab}\varphi^* x^a \wedge d\varphi^* x^b, \qquad S_1(\varphi) = \int_\Sigma k\varphi^*\Theta.$$

Note that the degrees of ω and Θ have been chosen so that $S(\varphi)$ is the integral of a $k+1$-form on the $k+1$-dimensional manifold Σ. If we consider the space of fields $Mor(T[1]\Sigma, \mathcal{M})$ as a graded manifold itself, then it has a natural structure of symplectic graded manifold and the action functional S satisfies $\{S, S\} = 0$. As we will see in Section 3.4, this means that $\{S, \cdot\}$ is a differential on the sheaf of functions of $Mor(T[1]\Sigma, \mathcal{M})$, giving rise to cohomology groups. The cohomology in degree 0 can be identified with the space of functions over solutions to the Euler-Lagrange equation of the field theory, so we obtain a resolution of this space.

The main reason why the AKSZ formalism is used in the literature is that the fact that $Mor(T[1]\Sigma, \mathcal{M})$ can be seen as a symplectic graded manifold where the action functional S satisfies $\{S, S\} = 0$ is extremely useful for the quantization of this model. Namely, this equation allows for a precise treatment of the **path integral** approach [9], which is a non-precise method for quantizing a field theory stating that the expectation of the measurement of $f \in C^\infty(F)$ for F the space of fields should be its expectation with respect to a measure μ_S defined on F as $\mu_S(\varphi) = exp(\frac{i}{\hbar}S(\varphi))\mu$ for μ a previously fixed measure on F. If $F = Mor(T[1]\Sigma, \mathcal{M})$, the expression $exp(\frac{i}{\hbar}S(\cdot))\mu$ makes sense as a differential form on F and $\{S, S\} = 0$ implies that it is an **integrable** differential form in the sense of integration on graded manifolds; hence, the path integral is well-defined.

As abstract as this language may seem, many important — and apparently distant at first sight — models are covered by this general theory, which also works particularly well for quantizing classical one-dimensional field theories with gauge symmetries. For example, topological Yang-Mills theory, the A-model, the B-model and the Courant σ-model arise as AKSZ models. The graded geometry of the NQ-manifold $\mathcal{M}$ plays an important role: symplectic submanifolds give higher analogs of metrics, the image under the fields φ of the boundary of Σ must lie on Lagrangian submanifolds of $\mathcal{M}$, etc.

In general, it can be said that a graded manifold is a generalization of an ordinary manifold which, using a familiar language, organizes the information of a complicated geometric structure at different *levels*, revealing itself as a powerful model for controlling different notions of symmetry (fermions interacting with bosons or physical fields interacting with ghost fields as transformations between coordinates of different parities or degrees) or morphism (homotopies as maps of degree 1, higher homotopies as maps of higher degree) between such objects.

1.3. Outline of this Work

This work is an introduction to the language of graded geometry with an eye toward its applications in generalized Riemannian geometry. We present the language of graded manifolds and we study the most important classes of examples of these. Then we prove Vaintrob's [53] and Roytenberg's [42] Theorems on classification of, respectively, Lie algebroids and Poisson manifolds and Courant algebroids as particular instances of graded manifolds. Finally, we apply this correspondence to the study of generalized geometry from the perspective of graded manifolds.

Chapter 2 serves as an introduction to graded geometry. We establish here once and for all the sign conventions that we shall use in this work, which we forewarn differ between authors. We present our definition of graded manifold and of all auxiliary objects such as vector bundles, vector fields or differential forms over graded manifolds, extending Cartan calculus to this setting. We also show some important examples which will be used in the rest of the work and we sketch some of the problems that arise when studying the space of morphisms between graded manifolds as a graded manifold itself.

Chapter 3 is an exposition of the different classes of graded manifolds that appear in the literature. Graded manifolds are usually a geometric model for complicated algebraic structures on what we can see as their sheaf of functions, and one common way in which these algebraic operations appear is through derived brackets. We study these in a purely algebraic way and then present a geometric definition of L_∞-algebras which relates to the algebraic one via derived brackets. L_∞-algebras are the local model for the so-called Q-manifolds which we define next; these appear in different ways in physical theories. We also discuss the extension of symplectic geometry to the graded setting, which seems to be crucial for quantizing σ-models based on graded manifolds. Finally, we study the structure of non-negatively graded manifolds; most of our examples belong to this class and here some of the complications of the graded world can be avoided. In particular, we prove Vaintrob's Theorem characterizing Lie algebroids as NQ-manifolds of degree 1.

Chapter 4 is devoted to the study of Courant algebroids as graded manifolds. First we introduce Courant algebroids and we show some important examples. Then we prove Ševera-Roytenberg's Theorem in two steps: first we characterize symplectic N-manifolds of degree 1 and 2 as, respectively, ordinary manifolds and pseudo-Euclidean vector bundles and then we characterize symplectic NQ-manifolds of degree 1 and 2 as, respectively, Poisson manifolds and Courant algebroids. By studying a Courant algebroid E in terms of its corresponding symplectic NQ-manifold $\mathcal{M}$ we show that deformations of E are encoded in the Q-cohomology of $\mathcal{M}$, we show how Dirac structures on E can be thought of as Q-Lagrangian submanifolds of $\mathcal{M}$ and we characterize these for the double of a Lie bialgebroid. Then we present the basic objects of generalized Riemannian geometry and we study them from the perspective of graded geometry. For this we construct, using a a generalized connection D on E, a graded Poisson manifold $\mathcal{M}^D$ with a $\Theta \in C^\infty(\mathcal{M}^D)$ satisfying $\{\Theta, \Theta\} = 0$ which is only isomorphic to $\mathcal{M}$ when E is transitive. This way we show how the equation $\{\Theta, \Theta\} = 0$ gives a Bianchi identity for the curvature of D and we construct a graded analog of a Morita equivalence from a generalized metric.

We have tried to present this theory as clearly as possible by introducing many examples in every section. Most of the results are given with very detailed proofs, and we give references for those that we do not prove. We have decided to avoid some of the technicalities of the graded setting which were not strictly required for the rest of the work or which we did not consider particularly meaningful for our purposes; namely, we do not review here the theory of integration on supermanifolds [36] or the more intricate sheaf-theoretic issues on the definition of graded manifolds [15], and we will not prove the classification theorem for smooth graded manifolds [3], [58].

CHAPTER 2
Graded Manifolds

In this chapter we present the basic definitions of graded geometry. In Section 2.1 we define **graded algebraic structures** and establish the sign conventions that will be used throughout the whole work. Then, in Section 2.2, we define **graded manifolds** and present the first examples of these. Section 2.3 is devoted to defining **vector fields** and **differential forms** on graded manifolds, and to extending Cartan calculus to this setting. Finally, in Section 2.4, we make some remarks on how to view the space of **morphisms** of graded manifolds as a graded manifold itself.

2.1. Graded Algebra

As it is always done in geometry, one must first begin by considering the algebraic notions underlying our objects of study. In this section we establish the **sign conventions** for many **graded algebraic structures** that will be extensively used in what follows. In general, we will study manifolds with a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading, but all the sign conventions and commutativity relations will depend exclusively on the $\mathbb{Z}/2\mathbb{Z}$-grading, which means that for these conventions it is enough to focus our attention on $\mathbb{Z}/2\mathbb{Z}$-graded algebraic structures.

Definition 2.1. *For any abelian group G, a G-**graded ring** is a ring R such that $R = \prod_{g \in G} R_g$ as groups and the subgroups R_g satisfy $R_g R_h \subset R_{gh}$. Elements from R_g are called **homogeneous of degree** g. If R is a $\mathbb{Z}/2\mathbb{Z}$-graded ring, elements of R_0 are called **even** and elements of R_1 are called **odd**, and the function assigning either 0 or 1 to homogeneous elements is the **parity** function, which we denote by p. A **morphism of G-graded rings** is a morphism of rings preserving the grading. A **supercommutative ring** is a $\mathbb{Z}/2\mathbb{Z}$-graded ring R with unit and such that $rs = (-1)^{p(r)p(s)} sr$ for homogeneous elements $r, s \in R$.*

If A is a graded ring, we will often write formulas involving the degrees of the elements $\forall x \in A$ when we mean *for every homogeneous element in A*. Since every element is a sum of homogeneous elements, these formulas can be applied to general elements by decomposing them in their homogeneous components. We will present the sign conventions from [29], which stick to the following general principle as much as possible: in any supercommutative algebraic structure, two adjacent homogeneous elements x, y can be interchanged if an additional term of $(-1)^{p(x)p(y)}$ is introduced. Thus, for a permutation $\sigma \in S_k$ we have

$x_1...x_k = (-1)^{\gamma(\sigma)} x_{\sigma(1)}...x_{\sigma(k)}$, where $(-1)^{\gamma(\sigma)}$ is the ***Koszul sign*** obtained by writing σ as a composition of transpositions and applying the above rule for each of them.

Given a supercommutative ring R and a left R-module E, a $\mathbb{Z}/2\mathbb{Z}$-grading on E is simply a splitting $E = E_0 \oplus E_1$. For such a grading, we can define a ***compatible*** right module structure on E by setting, for homogeneous elements $r \in R$ and $e \in E$, $r \cdot e := (-1)^{p(r)p(e)} e \cdot r$. Unless otherwise stated, we will simply write *R-module* for $\mathbb{Z}/2\mathbb{Z}$-graded R-modules with these left and right compatible actions. For example, the ***tensor product*** of two R-modules E_1, E_2 can be defined as usual, but noting that in this case we must require

$$(-1)^{p(r)p(e_1)} r \cdot e_1 \otimes e_2 = e_1 \cdot r \otimes e_2 = e_1 \otimes r \cdot e_2 = (-1)^{p(r)p(e_2)} e_1 \otimes e_2 \cdot r.$$

It follows that $E_1 \otimes_R E_2$ is also an R-module, with $p(e_1 \otimes e_2) = p(e_1) + p(e_2)$. Thus we can also define the ***tensor algebra*** of an R-module as the R-module $T(E) = \bigoplus_k E^{\otimes k}$, which has an additional $\mathbb{N}$-grading which we call ***homological degree*** and denote by d. Clearly, $T(E)$ is a ring and we can consider the double-sided ideals $I_\pm$ generated by $\{e_1 \otimes e_2 \pm (-1)^{p(e_1)p(e_2)} e_2 \otimes e_1 \,:\, e_1, e_2 \in E\}$. The quotients $S^*(E) := T(E)/I_-$ and $\Lambda^* E := T(E)/I_+$ are, respectively, the ***symmetric algebra*** and the ***exterior algebra*** of E and the induced products are denoted by $\odot$ and $\wedge$. While $S^*(E)$ is a supercommutative ring with the induced parity from the tensor product, elements of $\Lambda^* E$ satisfy the relation

$$\alpha \wedge \beta = (-1)^{p(\alpha)p(\beta) + d(\alpha)d(\beta)} \beta \wedge \alpha,$$

which means that $\Lambda^* E$ enters into a *new category* in which the sign rule is the one above, in the same way the exterior algebra of an (ordinary) commutative module is a supercommutative module.

Definition 2.2. *If E, F are R-modules, a map $l : E \to F$ is* **linear** *if $l(e \cdot r) = l(e) \cdot r$, $\forall e \in E, \forall r \in R$. We say that l is of parity ϵ if $p(l(e)) = p(e) + \epsilon$, $\forall e \in E$. In this case, note that $l(r \cdot e) = (-1)^{p(r)p(l)} r \cdot l(e)$. If $E_1, ..., E_k, F$ are R-modules, a map $m : E_1 \times ... \times E_k \to F$ is* **multilinear** *if*

$$m(e_1, ..., e_j \cdot r, e_{j+1}, ...e_k) = m(e_1, ..., e_j, r \cdot e_{j+1}, ...e_k) \quad j = 1, ..., k-1$$
$$m(e_1, ..., e_k \cdot r) = m(e_1, ..., e_k) \cdot r.$$

A multilinear map $m : E \times ... E \to F$ is **symmetric** *if*

$$m(e_1, ...e_j, e_{j+1}, ...e_k) = (-1)^{p(e_j)p(e_{j+1})} m(e_1, ..., e_{j+1}, e_j, , ...e_k)$$

and it is **skew-symmetric** *if*

$$m(e_1, ...e_j, e_{j+1}, ...e_k) = -(-1)^{p(e_j)p(e_{j+1})} m(e_1, ..., e_{j+1}, e_j, , ...e_k).$$

If E is an R-module, then the set of linear maps $l : E \to R$ is the ***dual*** of E, which we denote by $E^* := Hom_R(E, R)$. It is also an R-module with the grading from Definition 2.2 and the following action of R: $(r \cdot l)(e) = r \cdot l(e)$, $(l \cdot r)(e) = (-1)^{p(r)p(e)} l(e) \cdot r$. More generally, elements of $E_k^* \otimes_R ... \otimes_R E_1^*$ can be identified with multilinear maps $m : E_1 \times ... \times E_k \to R$ via

$$l_1 \otimes ... \otimes l_k(e_1, ..., e_k) = l_1(e_1)...l_k(e_k)(-1)^{e_1(l_2 + ... + l_k) + ... + e_{k-1} l_k},$$

where we have omitted writing the p to simplify the notation. Two important relations are

$$l_1 \otimes \dots \otimes l_k(e_1, \dots, e_k) = l_1 \otimes \dots \otimes l_j(e_1, \dots, e_j) l_{j+1} \otimes \dots \otimes l_k(e_{j+1}, \dots, e_k)(-1)^{(e_1+\dots+e_j)(l_{j+1}+\dots+l_k)} \tag{2.1}$$

and

$$\begin{aligned} l_j \otimes l_{j-1}(e_{j-1}, e_j) &= l_j(e_{j-1}) l_{j-1}(e_j)(-1)^{e_{j-1}l_{j-1}} = l_{j-1}(e_j) l_j(e_{j-1})(-1)^{e_{j-1}l_{j-1}+(l_j+e_{j-1})(l_{j-1}+e_j)} \\ &= l_{j-1}(e_j) l_j(e_{j-1})(-1)^{l_j(l_{j-1}+e_j)+e_{j-1}e_j} = (-1)^{l_j l_{j-1}+e_j e_{j-1}} l_{j-1} \otimes l_j(e_j, e_{j-1}). \end{aligned} \tag{2.2}$$

Elements of $S^*(E^*)$ and $\Lambda^* E^*$ can be identified with symmetric and skew-symmetric multilinear maps by summing over all their equivalence class. That is, we define

$$\begin{aligned} l_1 \odot \dots \odot l_k(e_1, \dots, e_k) &= \sum_{\sigma \in S_k} (-1)^{\gamma(\sigma)}\, l_{\sigma(1)} \otimes \dots \otimes l_{\sigma(k)}(e_1, \dots, e_k), \\ l_1 \wedge \dots \wedge l_k(e_1, \dots, e_k) &= \sum_{\sigma \in S_k} (-1)^{\gamma(\sigma)}\, sgn(\sigma)\, l_{\sigma(1)} \otimes \dots \otimes l_{\sigma(k)}(e_1, \dots, e_k), \end{aligned}$$

where $sgn(\sigma)$ is the sign of the permutation σ and $(-1)^{\gamma(\sigma)}$ is the Koszul sign subject to σ and the parities of the l_j's. The fact that these are in fact symmetric and skew-symmetric linear maps follows from Equations (2.1) and (2.2), which imply that the above formulas can also be written as

$$\begin{aligned} l_1 \odot \dots \odot l_k(e_1, \dots, e_k) &= \sum_{\sigma \in S_k} (-1)^{\gamma(\sigma)}\, l_1 \otimes \dots \otimes l_k(e_{\sigma(1)}, \dots, e_{\sigma(k)}), \\ l_1 \wedge \dots \wedge l_k(e_1, \dots, e_k) &= \sum_{\sigma \in S_k} (-1)^{\gamma(\sigma)}\, sgn(\sigma)\, l_1 \otimes \dots \otimes l_k(e_{\sigma(1)}, \dots, e_{\sigma(k)}), \end{aligned}$$

where the Koszul sign $(-1)^{\gamma(\sigma)}$ now depends on σ and the parities of the e_j's. For the sake of completeness we note that the above remarks imply that, for $\alpha \in \Lambda^p E^*$ and $\beta \in \Lambda^q E^*$,

$$\begin{aligned} \alpha \wedge \beta(e_1, \dots, e_{p+q}) &= \sum_{\sigma \in S_{p,q}} \alpha(e_{\sigma(1)}, \dots, e_{\sigma(p)}) \beta(e_{\sigma(p+1)}, \dots, e_{\sigma(p+q)})(-1)^{\beta(e_{\sigma(1)}+\dots+e_{\sigma(p)})}(-1)^{\gamma(\sigma)}\, sgn(\sigma) \\ &= (-1)^{\alpha\beta} \sum_{\sigma \in S_{p,q}} \beta(\alpha(e_{\sigma(1)}, \dots, e_{\sigma(p)}) e_{\sigma(p+1)}, \dots, e_{\sigma(p+q)})(-1)^{\gamma(\sigma)}\, sgn(\sigma). \end{aligned} \tag{2.3}$$

where $S_{p,q}$ is the set of $p+q$-permutations such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$, and similarly for the symmetric product. Finally, we remark that an element $\alpha \in S^k(E^*)$ can also be thought of as a degree k homogeneous polynomial $\alpha : E \to R$, acting as

$$\alpha(e) := \frac{1}{k!}\alpha(e, \dots, e).$$

This definition makes sense at least for $e \in E_0$ because $|S_{p,q}| = \binom{p+q}{q}$ and so, for $\alpha \in S^p(E^*)$ and $\beta \in S^q(E^*)$, we have

$$\alpha(e)\beta(e) = \frac{1}{p!}\alpha(e, \dots, e)\frac{1}{q!}\beta(e, \dots, e) = \frac{1}{(p+q)!}\binom{p+q}{q}\alpha(e, \dots, e)\beta(e, \dots, e) = (-1)^{\beta p e}\alpha \odot \beta(e).$$

Thus when $p(e) = 0$ this definition does indeed reflect the evaluation of a polynomial.

Definition 2.3. *Given a supercommutative ring R, a function $X : R \to R$ is a* **derivation** *of R if $X(fg) = X(f)g + (-1)^{p(f)p(g)}X(g)f$. We say that $p(X) = \epsilon$ if $p(X(f)) = \epsilon + p(f)$, $\forall f \in R$. In this case, note that $X(fg) = X(f)g + (-1)^{p(X)p(f)}fX(g)$. The set of derivations of R is denoted by $Der\ R$. The* **commutator** *of two derivations $X, Y \in Der\ R$ is the derivation $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.*

***Remark* 2.4.** *$Der\ R$*, is an R-module with the above grading and the action $(r \cdot X)(s) = r \cdot X(s)$ and $(X \cdot r)(s) = (-1)^{p(r)p(s)}X(s) \cdot r$. For $X \in Der\ R$, a quick computation shows

$$X(f_1 \cdot ... \cdot f_n) = X(f_1)f_2...f_n + (-1)^{f_2 f_1}X(f_2)f_1...f_n + ... + (-1)^{f_n(f_1+...+f_{n-1})}X(f_n)f_1...f_{n-1}.$$

In particular, for any R-module E we can consider the supercommutative ring $S^*(E^*)$. An element $e \in E$ induces a canonical derivation $\iota_e \in Der\ S^*(E^*)$ acting on elements $l \in E^*$ as $\iota_e(l) = (-1)^{p(l)p(e)}l(e)$. It extends to the whole of $S^*(E^*)$ as

$$\begin{aligned}\iota_e(l_1 \odot ... \odot l_p) = (-1)^{l_1 e}l_1(e)l_2 \odot ... \odot l_p + (-1)^{l_2 e + l_1 l_2}l_2(e)l_1 \odot ... \odot l_p \\ + ... + (-1)^{l_p e + l_p(l_1+...+l_{p-1})}l_p(e)l_1 \odot ... \odot l_{p-1}.\end{aligned}$$

Notice

$$\begin{aligned}&(-1)^{e(l_1+...+l_p)}\,\iota_e(l_1 \odot ... \odot l_p)(e_1, ..., e_{p-1}) \\ &\quad = l_1(e)l_2 \odot ... \odot l_p(e_1, ..., e_{p-1})(-1)^{e(l_2+...+l_p)} \\ &\qquad + l_2(e)l_1 \odot ... \odot \hat{l}_2 \odot ... \odot l_p(e_1, ..., e_{p-1})(-1)^{e(l_1+...+\hat{l}_2+...+l_p)}(-1)^{l_1 l_2} + ... \\ &\qquad + l_p(e)l_1 \odot ... \odot l_{p-1}(e_1, ..., e_{p-1})(-1)^{e(l_1+...+l_{p-1})}(-1)^{l_p(l_1+...+l_{p-1})} \\ &\quad = l_1 \odot ... \odot l_p(e, e_1, ..., e_{p-1}),\end{aligned}$$

(as usual, $\hat{l}_2$ means that l_2 does not appear in the corresponding term) so we can write $\iota_e \alpha = (-1)^{p(\alpha)p(e)}\alpha(e, \cdot, ..., \cdot)$ in general. When E is a purely odd vector space, this means $\iota_e \alpha = \alpha(\cdot, ..., \cdot, e)$.

Definition 2.5. *A* **Lie superalgebra of parity** *ϵ (or* **even/odd Lie superalgebra***) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V with a bilinear operation $[\cdot, \cdot] : V \otimes V \to V$ satisfying:*

1. $p([X, Y]) = p(X) + p(Y) + \epsilon$ *for* $X, Y \in V$,
2. $[X, Y] = -(-1)^{(p(X)+\epsilon)(p(Y)+\epsilon)}[Y, X]$ *for* $X, Y \in V$,
3. $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(p(X)+\epsilon)(p(Y)+\epsilon)}[Y, [X, Z]]$ *for* $X, Y, Z \in V$.

If $[\cdot, \cdot]$ fails to satisfy Property 2, we say that V is a **Loday superalgebra**. *If, in addition to Properties 1, 2 and 3, V has a super commutative product satisfying*

$$[X, YZ] = [X, Y]Z + (-1)^{p(Y)(\epsilon+p(X))}Y[X, Z],$$

we say that V is an **even/odd Poisson superalgebra**. *Odd Poisson superalgebras are also called* **Gerstenhaber algebras**.

Since we have defined a Lie superalgebra as a vector space and not just an R-module, it is assumed in Proposition 2.6 that R contains $\mathbb{R}$ as a subring, which will be the case in all the rings that we shall consider in this work.

***Proposition* 2.6.** The commutator $[\cdot,\cdot]$ endows $Der\ R$ with a structure of even Lie superalgebra structure which also satisfies

$$[X, fY] = X(f)Y - (-1)^{p(Y)(p(X)+p(f))} f[X,Y] \quad \text{for} \quad X, Y \in Der\ R \quad \text{and } f \in R$$

Proof.

A direct computation analogous to the ordinary one but using the appropriate sign rules. □

We emphasize again that we will consider $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded structures (rings, modules, Lie algebras, ...) but the $\mathbb{Z}$-grading will not play any role in the sign rules, as it will essentially be just a way of classifying elements in the pertinent set. Notice moreover that, if E is a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded R-module, then E^*, $S^*(E)$, $\Lambda^* E$,... are also $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded R-modules for the same reasons as in the plain $\mathbb{Z}/2\mathbb{Z}$-grading. We finish this section by introducing a notation that will be extensively used: If E is a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded R-module, then $E[k,\epsilon]$ is the $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded R-module with the same underlying set as E and grading given by

$$w_{E[k,\epsilon]}(e) = w_E(e) - k \qquad p_{E[k,\epsilon]}(e) = p_E(e) - \epsilon,$$

where $w_{E[k,\epsilon]}$ denotes the $\mathbb{Z}$-grading on $E[k,\epsilon]$, $p_{E[k,\epsilon]}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-grading on $E[k,\epsilon]$ and similarly for w_E, p_E. In particular, notice that a linear map $l : E \to F$ with $w(l(e)) = w(e) + k$ and $p(l(e)) = p(e) + \epsilon$ is the same as a morphism of graded R-modules (which is required to respect the grading) $l : E \to F[k,\epsilon]$. This means that $(E[k,\epsilon])^* = E^*[-k,\epsilon]$. We also write $\Pi E := E[0,1]$ and, if $\epsilon = k \mod 2$, $E[k,\epsilon] = E[k]$. This way, $\Pi E[k] = E[k,\epsilon]$ when $\epsilon = k+1 \mod 2$.

***Remark* 2.7.** Given an R-module E, the **décalage isomorphism** is the map

$$S^n(E[-1]) \to (\Lambda^n E)[-n]$$
$$e_1 \odot \ldots \odot e_n \mapsto (-1)^{\epsilon} e_1 \wedge \ldots \wedge e_n$$

with $\epsilon = \sum_{i=1}^n (n-i)(deg(e_i))$, where $deg(e_i)$ denotes the original grading of e_i on E. This map is an isomorphism of graded vector spaces [28], which shows that changing the parity of an R-module we may think of skew-symmetric maps as symmetric maps, and viceversa. This allows to define a super-commutative algebra structure on $\bigoplus_{n\geq 0}(\Lambda^n E)[-n]$ through the one in $S^*(E[-1])$. That is, the vector space $\Lambda^* E$ admits two non-isomorphic structures of algebra:

1. The one that we defined above, in which $\Lambda^* E$ has the $\mathbb{Z}/2\mathbb{Z}$-grading p that is obtained by summing the parities of the components and an additional $\mathbb{Z}$-grading d called homological degree; it satisfies $\alpha\wedge\beta = (-1)^{p(\alpha)p(\beta)+d(\alpha)d(\beta)}\beta\wedge\alpha$.

2. The one induced from the décalage isomorphism, which essentially accounts for defining a grading as $w(\alpha) = p(\alpha) + d(\alpha)$ and imposing super-commutativity with respect to w; it satisfies $\alpha\wedge\beta = (-1)^{(p(\alpha)+d(\alpha))(p(\beta)+d(\beta))}\beta\wedge\alpha$.

Some authors (as in [10]) prefer working with this second structure in order to remain in the super commutative framework. However, we prefer to stick with the structure arising from the tensor product because it allows us to think of elements in $\Lambda^* E$ as skew-symmetric forms in a natural way, which the other convention does not.

2.2. Graded Manifolds

In this section we define **graded manifolds** and present some basic examples. We will give a sheaf-theoretic definition of graded manifolds, completely analogous to the sheaf-theoretic construction of ordinary manifolds. The language of graded manifolds may seem at first sight like a pedantic way of approaching already known objects but, as we will see throughout this work, this approach allows for an intuitive geometric language which reveals new facets of the object of study and is usually powerful for unifying distant theories. We start with a motivating idea.

Consider the tangent space TM of an ordinary differentiable manifold M but assume that we want to see it as a geometric object — call it ΠTM — in which the fibers T_pM are purely odd vector spaces. What should we call then *functions* on ΠTM? In the ordinary setting, at each fiber T_pM functions are defined as (a completion of) the space $S^*(T_pM^*)$ of polynomials on T_pM, so the natural way to proceed is to define the structure of ΠTM in such a way that the space of functions at each fiber ΠT_pM is $S^*(\Pi T_pM^*) \cong \Lambda^* T_pM^*$ (this isomorphism is the décalage isomorphism from Remark 2.7). That is, the sheaf of functions on ΠTM should be $\Omega(M)$, the sheaf of differential forms on M. A way to see ΠTM as an object *similar* to a manifold but in which the right notion of function is an element of $\Omega(M)$ is the following:

Suppose $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is an atlas on M, so $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ are diffeomorphisms. Consider the algebras $C^\infty(\mathcal{V}_\alpha) := C^\infty(V_\alpha) \otimes \Lambda^*\mathbb{R}^n$, which can be identified with $\Omega(U_\alpha)$ via φ_α. The transition morphisms $\varphi_\alpha \circ \varphi_\beta^{-1}$ of M with Jacobian matrix $D_{\alpha,\beta}$ give transition morphisms for TM as $\varphi_\alpha \circ \varphi_\beta^{-1} \otimes D_{\alpha,\beta} : C^\infty(V_\alpha) \otimes \mathbb{R}^n \to C^\infty(V_\alpha) \otimes \mathbb{R}^n$. These extend in a unique way to morphisms of algebras $\psi_{\alpha,\beta} : C^\infty(\mathcal{V}_\beta) \to C^\infty(\mathcal{V}_\alpha)$ and in fact $\Omega(M) \cong \{\{f_\alpha\}_\alpha \in \prod_\alpha C^\infty(\mathcal{V}_\alpha) : f_\alpha = \psi_{\alpha,\beta} f_\beta\}$. This construction of $\Omega(M)$ is identical to the construction of the sheaf of C^∞ functions on M (not on TM), except that we are pulling back the algebras $C^\infty(\mathcal{V}_\alpha)$ instead of just $C^\infty(V_\alpha)$. The pair $(M, \Omega(M))$ can be seen through this construction as an object which is similar to a manifold and which we can call ΠTM.

The above example shows that the kind of objects that we want to study should have graded domains, as defined below, as their local model. The most common and elegant way to then glue the local pieces together involves the use of locally ringed spaces. In what follows *graded* means $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-*graded* unless otherwise stated, and all imposed commutativity relations are considered with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading, called ***parity*** and denoted by p as in Section 2.1. The $\mathbb{Z}$-grading is called ***weight*** and it is denoted by w.

Definition 2.8. *A* **graded domain** *is a pair* $\mathcal{V}_\alpha = (V_\alpha, C^\infty(\mathcal{V}_\alpha))$*, where* $V_\alpha \subset \mathbb{R}^{n_{0,0}}$ *is some open subset and* $C^\infty(\mathcal{V}_\alpha) = C^\infty(V_\alpha) \otimes A$ *for* A *a free* $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*-graded supercommutative* $\mathbb{R}$*-algebra finitely generated by elements of non-zero degree. The* **dimension on degree** $(k, \epsilon) \in Z \times \mathbb{Z}/2\mathbb{Z}$ *for* $(k, \epsilon) \neq (0, 0)$ *is the number of algebraically independent generators of* A *of that degree, and the dimension on degree* $(0, 0)$ *is* $n_{0,0}$.

In particular, a graded domain is a ringed space. Moreover, there is an analog of Hadamard's Lemma for graded manifolds implying that graded domains are in fact **locally ringed**

spaces; that is, the localization

$$C^\infty(\mathcal{V}_\alpha)_{|x} := \varinjlim_{V} C^\infty(V) \otimes A$$

at each point $x \in V_\alpha$, where V runs over $\{V \subset V_\alpha : V \text{ open}, x \in V\}$ and the direct limit is taken with respect to the restriction morphisms of $C^\infty(V_\alpha)$, is a local ring. See for example [15].

Definition 2.9. *A* **morphism of graded domains** *is a morphism of locally ringed spaces $\psi_{\alpha,\beta} : \mathcal{V}_\alpha \to \mathcal{V}_\beta$ such that the underlying morphisms of algebras $\psi^*_{\alpha,\beta} : C^\infty(\mathcal{V}_\beta) \to C^\infty(\mathcal{V}_\alpha)$ preserve the grading.*

Definition 2.10. *A* **graded manifold** *is a locally ringed space $\mathcal{M} = (M, C^\infty(\mathcal{M}))$, where M is a Hausdorff, second countable topological space and $C^\infty(\mathcal{M})$ is a sheaf of graded algebras over M such that*

1. *There exists a covering U_α of M and isomorphisms of locally ringed spaces $\varphi_\alpha : \mathcal{U}_\alpha := (U_\alpha, C^\infty(\mathcal{M})_{|U_\alpha}) \to \mathcal{V}_\alpha$ to some graded domains $\mathcal{V}_\alpha$.*

2. *The* ***transition morphisms*** *$\varphi_\alpha \circ \varphi_\beta^{-1}$ are isomorphisms of graded domains.*

The **dimension** *of $\mathcal{M}$ on each degree $(k, \epsilon) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the dimension of any of these graded domains in the corresponding degree. We write $C^\infty_{k,\epsilon}(\mathcal{M})$ for the (k,ϵ)-degree part of $C^\infty(\mathcal{M})$. A* **morphism of graded manifolds** *is a morphism of locally ringed spaces whose pull-backs preserve the grading. An* **open subset** *of $\mathcal{M}$ is a graded manifold $\mathcal{U} = (U, C^\infty(\mathcal{M})_{|U})$ for an open subset $U \subset M$, and we write $\mathcal{U} \subset \mathcal{M}$.*

We will sometimes use the word **supermanifold** for a graded manifold with trivial $\mathbb{Z}$-grading; this is Berezin-Kostant-Leite's approach to supermanifolds as presented, for example, in [36], [34]. This seems to be the most common way to proceed in recent work, although some authors follow de Witt [13] and consider a different category of supermanifolds based on super-Euclidean space with super-functions of super-numbers as a local model. See [4] for a proof that these two approaches are equivalent and [5] for a further comparison with other similar notions of supermanifold.

If $\mathcal{M} = (M, C^\infty(\mathcal{M}))$ is a graded manifold, then $(M, C^\infty(M))$ is an ordinary manifold, where $C^\infty(M) = C^\infty(\mathcal{M})/I$, and I is the ideal generated by non-zero degree elements. On coordinate domains U, functions of $\mathcal{M}$ can be locally written as

$$f = \sum_\alpha f_\alpha \xi^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ runs over arbitrarily large multi-indices, $f_\alpha \in C^\infty(U)$ are ordinary C^∞ functions, $\{\xi_1, \dots, \xi_d\}$ is a set of algebraically independent generators of the model algebra A and $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$. In principle, these are formal sums that can have an infinite number of terms, so some care must be taken with constructions such as tensor products. The following example shows why we need to consider formal power series.

Example* 2.11 *(Formal Power Series are Necessary). Consider two graded domains $\mathcal{V}_\alpha$, $\mathcal{V}_\beta$ with dimension 1 on degrees $(0,0)$, $(1,0)$ and $(-1,0)$. Then, the following is an admissible morphism

$$\begin{aligned} \psi^*_{\alpha,\beta} : C^\infty(\mathcal{V}_\beta) &\to C^\infty(\mathcal{V}_\alpha) \\ x_\beta &\mapsto x_\alpha + \xi_\alpha \eta_\alpha \\ \xi_\beta &\mapsto \xi_\alpha \\ \eta_\beta &\mapsto \eta_\alpha, \end{aligned}$$

where $w(x_\alpha) = w(x_\beta) = 0$, $w(\xi_\alpha) = w(\xi_\beta) = 1$ and $w(\eta_\alpha) = w(\eta_\beta) = -1$. Thus, we must require that functions such as $\sin(x_\beta) \in C^\infty(\mathcal{V}_\beta)$ have an image on $C^\infty(\mathcal{V}_\alpha)$ which cannot be anything different from the formal power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x_\alpha + \xi_\alpha \eta_\alpha)^{2n+1}.$$

This example also shows that one cannot define a category of C^k graded manifolds in this way unless some additional assumptions on the degrees are included.

Despite anomalies as the one in Example 2.11, there are many situations in which formal power series can be avoided. For example, if one is only interested on algebraic functions (i.e., polynomials) on degree 0, these problems disappear. Also, if all ξ_j's are odd, then the above sums will all be finite because every ξ^α is idempotent. Finally, if there are no negatively weighted (or no positively weighted) coordinates, requiring that the model algebra A consist only on finite sums will not give any problems because we cannot obtain a degree 0 coordinate from non-zero degree coordinates; most of our graded manifolds will fit into this category, which will be studied in more detail in Section 3.5. However, grading in both positive and negative degrees is important in the physics literature, as it appears in the BRST formalism [9].

Example* 2.12 *(Shifted Vector Bundles). If $E \to M$ is some vector bundle of rank d, for each $(k,\epsilon) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ we define the graded manifold $E[k,\epsilon] = (M, C^\infty(E[k,\epsilon]))$ by assigning degree (k,ϵ) to the fiber coordinates of E. This means that we construct $C^\infty(E[k,\epsilon])$ using a trivialization $\{U_\alpha\}_\alpha$ of E over M and gluing the algebras $C^\infty(U_\alpha) \otimes A$, where $A = \mathbb{R}[\xi_1, ..., \xi_d]$ for variables $\xi_1, ..., \xi_d$ of degree (k,ϵ), with the transition morphisms of E in the same way as we did for ΠTM at the beginning of this section. The notation is chosen so that, when $E = M \times V$ for a vector space V,

$$C^\infty(E[k,\epsilon]) = C^\infty(M) \otimes S^*((V[k,\epsilon])^*) = C^\infty(M) \otimes S^*(V^*[-k,\epsilon]),$$

where $V[k,\epsilon]$ is as defined in Section 2.1. In general, we obtain $C^\infty(E[k,\epsilon]) = \Gamma(S^*E^*)$ if $\epsilon = 0$ and $C^\infty(E[k,\epsilon]) = \Gamma(\Lambda^* E^*)$ if $\epsilon = 1$. If V is a vector space, we see it as a vector bundle over a point and define the graded manifold $V[k,\epsilon] = (\{*\}, S(V^*[-k,\epsilon]))$. In the special case of $E = TM$ or $E = T^*M$ we write $T[k,\epsilon]M$ instead of $TM[k,\epsilon]$, and similarly for the cotangent bundle. This will avoid confusion when we consider the tangent bundle of a graded manifold, as in Example 2.14 below.

Example 2.13 (Vector Bundles over Graded Manifolds). More generally, we can define ***vector bundles over a graded manifold*** $\mathcal{M}$ as graded manifolds $\mathcal{E}$ with a morphism of graded manifolds $p : \mathcal{E} \to \mathcal{M}$ such that there exists an open covering $\{\mathcal{U}_\alpha\}_\alpha$ of $\mathcal{M}$ and isomorphisms of graded algebras

$$\varphi_\alpha : p^*\left(C^\infty(\mathcal{U}_\alpha)\right) \to C^\infty(\mathcal{U}_\alpha) \otimes S^*W$$

for a graded vector space W (the **typical fibre**) satisfying, for $v \in W$,

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : C^\infty(\mathcal{U}_\beta) \otimes S^*W &\to C^\infty(\mathcal{U}_\alpha) \otimes S^*W \\ v &\mapsto A_{\alpha,\beta}(v) \end{aligned}$$

for $\mathbb{R}$-linear maps $A_{\alpha,\beta} : W \to C^\infty(\mathcal{U}_\alpha) \otimes W$ preserving the grading.

For a vector bundle $\mathcal{E} \to \mathcal{M}$ the sheaf of functions $C^\infty(\mathcal{E})$ is a $C^\infty(\mathcal{M})$-module, and the above morphisms still preserve the grading if we shift the degree and parity of W, so we can also define $\mathcal{E}[k, \epsilon]$ in the natural way. Constructions such as the direct sum, tensor product or dual of these vector bundles are also well-defined. The ***sheaf of sections*** of the vector bundle $\mathcal{E} \to \mathcal{M}$ is the subset $\Gamma(\mathcal{E}) \subset C^\infty(\mathcal{E}^*)$ of functions on $\mathcal{E}^*$ that are $C^\infty(\mathcal{M})$-linear on the fibers. Then ***morphisms of vector bundles*** are morphisms of graded manifolds that are required to preserve the linear structure; that is, $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ with $\varphi^*(\Gamma(\mathcal{E}_2^*)) \subset \Gamma(\mathcal{E}_1^*)$. The localization of the $C^\infty(\mathcal{M})$-module $\Gamma(\mathcal{E})$ at a point $p \in M$ is an A-module, for A the free supercommutative algebra over which $\mathcal{M}$ is modelled. These objects will play an important role when we discuss N-manifolds in Section 3.5 and thereafter.

If the $\mathbb{R}$-linear maps $A_{\alpha,\beta} : W \to C^\infty(\mathcal{U}_\alpha) \otimes W$ do not preserve the original grading but the vector space W is concentrated on a single degree (k, ϵ), there is still a way to assign a grading on the algebra $C^\infty(\mathcal{E})$. It consists simply on assigning weight zero to the variables on the base $\mathcal{M}$. In particular, this provides an additional $\mathbb{Z}$-grading on $\mathcal{E}$, for any $\mathcal{E}$ constructed as above.

Example 2.14 (Tangent and Cotangent Bundles of a Graded Manifold). If $\mathcal{M}$ is a graded manifold, we shall define its tangent and cotangent bundles as follows. Let $\{\mathcal{U}_\alpha\}_\alpha$ be an open cover of $\mathcal{M}$ with isomorphisms $\varphi_\alpha : C^\infty(\mathcal{U}_\alpha) \to C^\infty(V_\alpha) \otimes A$ for $V_\alpha \subset \mathbb{R}^n$ and $A = \mathbb{R}[\xi^1, ..., \xi^m]$. Then the changes of coordinates $\varphi_\alpha \circ \varphi_\beta^{-1} : C^\infty(V_\beta) \otimes A \to C^\infty(V_\alpha) \otimes A$ have a Jacobian matrix (taking formal derivatives on the coordinates of non-zero degree) $D_{\alpha,\beta} \in C^\infty(V_\beta) \otimes A \otimes GL(W)$, where $W := span\,\{v^1, ..., v^n, \theta^1, ..., \theta^m\}$ with $deg(v^a) = (0, 0)$, $deg(\theta^i) = deg(\xi^i)$. If $\{y^k\}_k$ denote coordinates on W of arbitrary degrees, it is easy to see the (i, j)th entry of $D_{\alpha,\beta}$ on these coordinates has degree $deg(y^i) - deg(y^j)$ because $\varphi_\alpha \circ \varphi_\beta^{-1}$ preserves the grading and taking ∂_{y^j} lowers the degree by $deg(y^j)$; this implies that $D_{\alpha,\beta}$ preserves the grading of W and that $(D_{\alpha,\beta}^{-1})^t$ preserves the grading of $W^* = span\,\{p_1, ..., p_n, \rho_1, ..., \rho_m\}$ with $deg(p_i) = (0, 0)$, $deg(\rho_i) = -deg(\xi_i)$.

The **tangent bundle** $T\mathcal{M}$ of $\mathcal{M}$ is the vector bundle over $\mathcal{M}$ with typical fibre W and transition functions $D_{\alpha,\beta}$ and the **cotangent bundle** $T^*\mathcal{M}$ of $\mathcal{M}$ is the vector bundle over $\mathcal{M}$ with typical fibre W^* and transition functions $(D_{\alpha,\beta}^{-1})^t$. This grading on $T^*\mathcal{M}$ is known in the physics literature as **ghost number**. According to the final remarks in Example 2.13,

$T\mathcal{M}$ and $T^*\mathcal{M}$ have an additional $\mathbb{Z}$-grading assigning degree 0 to base coordinates and degree 1 to fiber coordinates. In any case, these are vector bundles over a graded manifold and, as such, we can also define their shifts, which we denote by $T[k,\epsilon]\mathcal{M}$ and $T^*[k,\epsilon]\mathcal{M}$.

For example, given an ordinary vector bundle $E \to M$ we can consider the graded manifold $E[1]$ with coordinates $\{x^a, \xi^i\}$ with $p(x^a) = w(x^a) = 0$ and $p(\xi^i) = w(\xi^i) = 1$. Its tangent bundle is $TE[1]$ with coordinates $\{x^a, \xi^i, v^a, \theta^i\}$ with $p(v^a) = w(v^a) = 0$ and $p(\theta^i) = w(\theta^i) = 1$, while $T^*E[1]$ has coordinates $\{x^a, \xi^i, p_a, \rho_a\}$ with $p(p_a) = w(p_a) = 0$ and $p(\rho_i) = w(\rho_i) = -1$. The additional $\mathbb{Z}$-grading mentioned above is (for example, for $T^*E[1]$,) $w'(x^a) = w'(\xi^i) = 0$, $w'(p_a) = w'(\rho_a) = 1$. The graded manifold $T^*[2]E[1]$, which has the same coordinates as $T^*E[1]$ but with $p(p_a) = w(p_a) = 2$ and $p(\rho_i) = w(\rho_i) = 1$, will play an important role in the Ševera-Roytenberg correspondence.

Example 2.15 (Multiple Vector Bundles). Let $E \to M$ be an ordinary vector bundle and consider the graded manifold $T^*[2]E[1]$ with coordinates $(x^a, \xi^i, p_a, \rho_i)$, as in Example 2.14. Furthermore, consider $T^*[2]E^*[1]$ with coordinates $(x^a, \eta_i, p_a, \theta^i)$. The **Legendre transformation** is the canonical isomorphism of graded manifolds $\mathcal{L} : T^*[2]E[1] \to T^*[2]E^*[1]$ with pull-back $x^a \mapsto x^a$, $\eta_i \mapsto \rho_i$, $p^a \mapsto p^a$, $\theta^i \mapsto \xi^i$. It can also be described in an invariant way as follows.

For a curve $\gamma : I \to E \oplus E^*$, $\gamma(t) = (x(t), e(t) + \xi(t))$ we write $\gamma_E(t) = (x(t), e(t)) \in E$, $\gamma_{E^*}(t) = (x(t), \xi(t)) \in E^*$ and $ev(\gamma(t)) = \xi(t)(e(t)) \in \mathbb{R}$. Let $\gamma(0) = (x_0, e_0 + \xi_0) \in E \oplus E^*$ and let $[\gamma_E] \in T_{(x_0,e_0)}E$ be the tangent vector determined by γ_E (and similarly for $[\gamma_{E^*}]$). For each $(x_0, e_0, F) \in T^*E$, we claim that there exists a unique $(x_0, \xi_0, G) \in T^*E^*$ such that

$$F([\gamma]_E) + G([\gamma_{E^*}]) = \frac{d}{dt}_{|t=0} ev(\gamma(t))$$

for every $\gamma : I \to E \oplus E^*$ with $\gamma(0) = (x_0, e_0 + \xi_0) \in E \oplus E^*$. To prove the claim, we first note that taking γ with $[\gamma_{E^*}] = 0$ we obtain that $\xi_0(v) = F([e_0 + tv])$ necessarily, while taking γ with $[\gamma_E] = 0$ shows $G([\xi_0 + t\alpha]) = \alpha(e_0)$. It only remains to define G over horizontal vector fields on TE^*. For this we take a connection ∇ on E with dual connection ∇^* on E^*; these allow to lift a curve $x(t) \in M$ to parallel curves $(x(t), \Gamma_{e_0}(x(t))) \in E$ and $(x(t), \Gamma^*_{\xi_0}(x(t))) \in E^*$ and they satisfy $\frac{d}{dt}_{|t=0}\Gamma^*_{\xi_0}(x(t))(\Gamma_{e_0}(x(t))) = 0$. Hence, we are forced to define $G([\Gamma^*_{\xi_0}(x(t))]) = -F([\Gamma_{e_0}(x(t))])$. This completes the definition of G and, in fact, this definition does not depend on ∇: if G has been defined through ∇ while $\tilde{\Gamma}_{e_0}$, $\tilde{\Gamma}^*_{\xi_0}$ denote the parallel transport maps of $\tilde{\nabla}$, then $[\tilde{\Gamma}_{e_0}(x(t))] = [\Gamma_{e_0}(x(t))] + [e_0 + tv]$ and $[\tilde{\Gamma}^*_{\xi_0}(x(t))] = [\Gamma^*_{\xi_0}(x(t))] + [\xi_0 + t\alpha]$ for some $v \in E_p$, $\alpha \in E^*_p$, so

$$G([\tilde{\Gamma}^*_{\xi_0}(x(t))]) = -F([\Gamma_{e_0}(x(t))]) + \alpha(e_0) = -F([\tilde{\Gamma}_{e_0}(x(t))]) + \xi_0(v) + \alpha(e_0)$$

and $\xi_0(v) + \alpha(e_0) = \frac{d}{dt}_{|t=0}\tilde{\Gamma}^*_{\xi_0}(x(t))(\tilde{\Gamma}_{e_0}(x(t))) = 0$. Thus we can define the isomorphism $\mathcal{L} : T^*E \to T^*E^*$ by $(x_0, e_0, F) \to (x_0, \xi_0, G)$, which in local coordinates coincides with the one defined above.

In any case, what is important about $\mathcal{L}$ is that it induces a **double vector bundle** structure

$$\begin{tikzcd} T^*[2]E[1] \cong T^*[2]E^*[1] \arrow[r] \arrow[d] & E^*[1] \arrow[d] \\ E[1] \arrow[r] & M; \end{tikzcd}$$

i.e., every arrow is a vector bundle projection. The projection of $(x_0, e_0, F) \in T^*E$ onto E is (x_0, e_0) and its projection onto E^* is (x_0, ξ_0), as defined above. This structure gives a $\mathbb{Z} \times \mathbb{Z}$-grading on $T^*[2]E[1]$ by assigning degree 1 to fiber coordinates and degree 0 to base coordinates with respect to each of the fibrations, and our original grading is the sum of these two. Remarkably, T^*E is **not** a vector bundle over M, so this is one example in which the language of graded geometry helps us see T^*E as a geometric object *over* M. In general, one can define a **k-fold vector bundle** (or double, triple, etc. vector bundle) as a $\mathbb{Z}^k$-graded manifold such that all coordinates have either weight 0 or weight 1 on each of the gradings. As shown in [56], this gives multilinear changes of coordinates. See also [32] for a detailed exposition of double vector bundles.

Example* 2.16 *(Jet Bundle). If M is some ordinary manifold, the space of l-jets is the following fibre bundle. We define an equivalence relation on C^∞ paths $\gamma : (-\epsilon, \epsilon) \to M$ passing through $p \in M$ by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \frac{d^r}{dt^r}\gamma_1 = \frac{d^r}{dt^r}\gamma_2 \qquad r = 0, ..., l\,,$$

where the derivative is taken in $\mathbb{R}^n$ after using a coordinate chart around p. The quotient of all paths by this relation is the space $J^l_p M$ of l-jets of M at p, and the total space $J^l M$ of l-jets is the bundle which has the same trivializations as M and fibers $J^l_p M$. Of course, $J^1 M = TM$ is just the tangent space, but for $l \geq 2$ we do not obtain a vector bundle structure; namely because the rth derivative of $\lambda\gamma$ is λ^r times the rth derivative of γ. Thus, the vector bundle structure is substituted by an $\mathbb{R}^\times$-action on $J^l M$. This bundle can also be interpreted as a graded manifold, where coordinates representing rth derivatives have degree $(r, 0)$. This $\mathbb{R}^\times$ action appears in every graded manifold through the ***Euler vector field*** that we shall define in Section 2.3 and it is an interesting way of interpreting what the grading means. Of course, one can also define the space of jets of a graded manifold and its shifts with a similar construction as the one in Example 2.14.

These examples, and the definition itself of a graded manifold, may induce the idea that graded manifolds are just bundles over the base space M. In fact, Batchelor's Theorem [3] and its generalization to the graded setting — sketched in [58] — state that any graded manifold $\mathcal{M}$ is isomorphic (as a graded manifold) to a graded vector bundle over the base space M. However, this isomorphism is ***non canonical***, which reflects the main difference between bundles and graded manifolds: morphisms of graded manifolds are way less restrictive than those of bundles, giving rise to a very different category. The reason for this is illustrated by the following Example, which is based on the same idea as Example 2.11.

Example* 2.17 *(Graded Manifolds Are Not Bundles). Let $A = \mathbb{R}[\xi^1, \xi^2]$ for variables ξ^1, ξ^2 of degree $(0, 1) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and consider $\mathcal{M} = (\mathbb{R}, C^\infty(\mathbb{R}) \otimes A)$. Although $\mathcal{M}$

is clearly isomorphic to ΠE, for E a trivial vector bundle of rank 2 over $\mathbb{R}$, the following morphism $\varphi : \mathcal{M} \to \mathcal{M}$ does not correspond to any vector bundle morphism $E \to E$.

$$\begin{aligned} \varphi^* : C^\infty(\mathcal{M}) &\to C^\infty(\mathcal{M}) \\ x &\mapsto x + \xi^1\xi^2 \\ \xi^1 &\mapsto \xi^1 \\ \xi^2 &\mapsto \xi^2 \end{aligned}$$

The above formula does not produce a morphism in the vector bundle interpretation because it mixes *base* and *fiber* coordinates, but it does in the graded context because $\xi^1\xi^2$ is even. Notice that the same happens if we perform the same construction with ξ^1 of degree $(1, 0)$ and ξ^2 of degree $(-1, 0)$.

2.3. Vector Fields and Differential Forms

In this section we construct **vector fields** and **differential forms** on graded manifolds in a purely algebraic way, which is the way in which we will see them in this work. We also develop the useful tools of Cartan calculus for graded manifolds.

Definition 2.18. *Given a graded manifold $\mathcal{M}$, we consider the sheaf of $C^\infty(\mathcal{M})$-modules $(M, Der\, C^\infty(\mathcal{M}))$. A* **vector field** *on $\mathcal{U} \subset \mathcal{M}$ is an element of $Der\, C^\infty(\mathcal{U})$, and a* **differential p-form** *on $\mathcal{U} \subset \mathcal{M}$ is an element of $\Omega^p(\mathcal{U}) := \Lambda^p(Der\, C^\infty(\mathcal{U}))^*$.*

***Remark* 2.19.** As usual, vector fields on $\mathcal{M}$ can be identified with $\Gamma(T\mathcal{M})$ and 1-forms on $\mathcal{M}$ can be identified with $\Gamma(T^*\mathcal{M})$, where the tangent and cotangent bundles of $\mathcal{M}$ are as defined in Example 2.14.

As we did in Section 2.1, we can define a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading for vector fields which makes $Der\, C^\infty(\mathcal{M})$ a sheaf of Lie superalgebras. Differential forms have, in addition to the $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading induced by the one on $Der\, C^\infty(\mathcal{M})$, an additional $\mathbb{Z}$-grading given by the homological degree and denoted by d. Recall that, for $\alpha, \beta \in \Omega(\mathcal{M}) = \prod_{p\in\mathbb{N}} \Omega^p(\mathcal{M})$, we have $\alpha \wedge \beta = (-1)^{p(\alpha)p(\beta)+d(\alpha)d(\beta)}\beta \wedge \alpha \in \Omega^{d(\alpha)+d(\beta)}(\mathcal{M})$. If $\{\xi^j\}_{j=1}^n$ are local coordinates (with arbitrary weights and parities) on $\mathcal{U} \subset \mathcal{N}$, any vector field $X \in Der\, C^\infty(\mathcal{U})$ can be written as

$$X = \sum_{j=1}^n f_j \frac{\partial}{\partial \xi^j}$$

for some functions $f_j = (-1)^{p(\xi^j)}X(\xi^j) \in C^\infty(\mathcal{U})$, where the derivations $\frac{\partial}{\partial \xi^j}$ are defined by $\frac{\partial}{\partial \xi^j}(\xi^i) = (-1)^{p(\xi^j)}\delta_{ij}$ and extended through Leibniz's rule. Similarly, a differential p-form $\alpha \in \Omega^p(\mathcal{U})$ can be written as

$$\alpha = \sum_{i_1+\ldots+i_n=p} g_{i_1\ldots i_d}(d\xi^1)^{i_1} \wedge \ldots \wedge (d\xi^d)^{i_n}$$

for some functions

$$g_{i_1\ldots i_n} = (-1)^{p+1}\alpha(\frac{\partial}{\partial \xi^n}, \overset{i_n}{\ldots}, \frac{\partial}{\partial \xi^n}, \ldots, \frac{\partial}{\partial \xi^1}, \overset{i_1}{\ldots}, \frac{\partial}{\partial \xi^1}),$$

where the 1-forms $d\xi^j$ are defined by $d\xi^j(\frac{\partial}{\partial \xi^i}) = \delta_{ij}$ and $(d\xi^j)^{i_j} := d\xi^j \wedge \overset{i_j}{\dots} \wedge d\xi^j$. We will often write this as

$$\alpha = \frac{1}{p!} \sum_{j_1,\dots,j_p=1}^{d} g_{j_1,\dots,j_p} d\xi^{j_1} \dots d\xi^{j_p},$$

In particular, notice that

$$\begin{aligned} w(\frac{\partial}{\partial \xi^j}) &= -w(\xi^j), & p(\frac{\partial}{\partial \xi^j}) &= p(\xi^j), \\ w(d\xi^j) &= w(\xi^j), & p(d\xi^j) &= p(\xi^j), \\ w(f_j) &= w(X) + w(\xi^j), & p(f_J) &= p(X) + p(\xi^j), \\ w(g_{i_1\dots i_d}) &= w(\alpha) - \sum_{j=1}^{d} i_j w(\xi^j), & p(g_{i_1\dots i_d}) &= p(\alpha) + \sum_{j=1}^{d} i_j p(\xi^j). \end{aligned}$$

***Remark* 2.20.** Recall Remark 2.7. If $\Omega(\mathcal{M}) = \Lambda^*(Der\, C^\infty(M))^*$ is considered with the algebra structure arising from the décalage isomorphism, then there exists an isomorphism of algebras $\Omega(\mathcal{M}) \cong C^\infty(T[1]\mathcal{M})$ (see Example 2.14), just as in the ordinary case. However, we will always consider $\Omega(\mathcal{M})$ with the algebra structure coming from the tensor product because otherwise we would not be able to interpret differential forms as skew-symmetric maps in any natural way.

In Section 2.1 we defined derivations of supercommutative rings, but $\Omega(M)$ is not supercommutative. However, it satisfies a similar property and, accordingly, we can define a ***derivation of*** $\Omega(\mathcal{M})$ as an operator $X : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ satisfying $X(\alpha \wedge \beta) = X(\alpha) \wedge \beta + (-1)^{p(\alpha)p(\beta)+d(\alpha)d(\beta)} X(\beta) \wedge \alpha$. If $w(X(\alpha)) = k + w(\alpha)$, $p(X(\alpha)) = \epsilon + p(\alpha)$ and $d(X(\alpha)) = p + d(\alpha)$ for every $\alpha \in \Omega(\mathcal{M})$, we say that X is homogeneous of degree (k, ϵ, p). Notice that, in this case, $X(\alpha \wedge \beta) = X(\alpha) \wedge \beta + (-1)^{p(\alpha)p(X)+d(\alpha)d(X)} \alpha \wedge X(\beta)$. With the commutator $[X, Y] = X \circ Y - (-1)^{p(X)p(Y)+d(X)d(Y)} Y \circ X$, $Der\, \Omega(\mathcal{M})$ satisfies Lie algebra properties as the ones in Definition 2.5, but changing $p(X)p(Y)$ by $p(X)p(Y) + d(X)d(Y)$.

Definition 2.21. *The* **differential of a function** $f \in C^\infty(\mathcal{M})$ *is the* 1*-form* df *acting as* $df(X) = (-1)^{p(X)p(f)} X(f)$. *Notice that* $w(df) = w(f)$ *and* $p(df) = p(f)$. *The* **exterior derivative** d *is the only degree* $(0, 0, 1)$ *derivation of* $\Omega(\mathcal{M})$ *satisfying* $d(f) = df$ *for* $f \in \Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$ *and* $d^2 = 0$. *Given a vector field* X*, the* **contraction by** X *is the operator* $\iota_X : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ *defined by* $\iota_X \alpha(X_1, \dots, X_{p-1}) = (-1)^{p(X)p(\alpha)} \alpha(X, X_1, \dots, X_{p-1})$ *for* $\alpha \in \Omega^p(\mathcal{M})$. *It is a derivation of degree* $(w(X), p(X), -1)$. *The* **Lie derivative** *with respect to* X *is* $\mathcal{L}_X := [d, \iota_X] = d\iota_X + \iota_X d$, *which is a derivation of* $\Omega(\mathcal{M})$ *of degree* $(w(X), p(X), 0)$.

***Proposition* 2.22.** The above objects are well-defined. Moreover, the following properties are satisfied.

1. $\mathcal{L}_X f = \iota_X(df) = X(f)$.

2. $[d, \mathcal{L}_X] = 0$.

3. $[\iota_X, \iota_Y] = 0$.

4. $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$.

5. $[\mathcal{L}_X, \mathcal{L}_Y] = -\mathcal{L}_{[Y,X]}$.

6. $d\alpha(X_0, ..., X_n) = \sum_j X_j(\alpha(X_0, ..., \hat{X}_j, ..., X_n))(-1)^{X_j(\alpha+X_0+...+X_{j-1})+j}$
$+ \sum_{i<j} \alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n)(-1)^{X_i(X_0+...+X_{i-1})+X_j(X_0+...+\hat{X}_i+...+X_{j-1})+i+j}$.

Proof.

$\Omega(\mathcal{M})$ is locally generated as an algebra by functions $f \in C^\infty(\mathcal{U})$ and exact forms $df \in \Omega^1(\mathcal{U})$. Since derivations of $\Omega(M)$ are required to satisfy Leibniz's rule, they are determined by their action on these generators. In particular, d is well-defined. The computation that shows that ι_X is indeed a derivation is the same as in Remark 2.4.

Now 1 and 2 are immediate and 3 means simply that forms are skew-symmetric. To see 4, it suffices to prove it for exact forms and in that case

$$\begin{aligned}[\mathcal{L}_X, \iota_Y](df) &= (\iota_X d + d\iota_X)\iota_Y df - (-1)^{p(X)p(Y)}\iota_Y(\iota_X d + d\iota_X)df \\ &= \iota_X d(\iota_Y df) - (-1)^{p(X)p(Y)}\iota_Y d(\iota_X df) = \iota_X d(Y(f)) - (-1)^{p(X)p(Y)}\iota_Y d(X(f)) \\ &= XY(f) - (-1)^{p(X)p(Y)}YX(f) = [X,Y](f) = \iota_{[X,Y]}df.\end{aligned}$$

Then 5 follows from the Jacobi identity

$$[\mathcal{L}_X, [d, \iota_Y]] = [[\mathcal{L}_X, d], \iota_Y] + (-1)^{p(X)p(Y)}[d, [\mathcal{L}_X, \iota_Y]] = (-1)^{p(X)p(Y)}[d, \iota_{[X,Y]}] = -\mathcal{L}_{[Y,X]}$$

The way to prove 6 is by noting that this formula does indeed define a $C^\infty(\mathcal{M})$-linear form:

$$\begin{aligned}&X_j(\alpha(X_0, ..., fX_i, ..., \hat{X}_j, ..., X_n))(-1)^{X_j(\alpha+X_0+...+X_i+f+...+X_{j-1})+j} \\ &\quad= \big(X_j(f)\alpha(X_0, ..., X_i, ..., \hat{X}_j, ..., X_n) + (-1)^{X_j f} fX_j(\alpha(X_0, ..., X_i, ..., \hat{X}_j, ..., X_n))\big) \\ &\quad\times (-1)^{X_j(\alpha+X_0+...+X_{j-1})+f(\alpha+X_0+...+X_{i-1})+X_j f+j}\end{aligned}$$

$$\begin{aligned}&\alpha([fX_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n)(-1)^{(f+X_i)(X_0+...+X_{i-1})+X_j(X_0+...+\hat{X}_i+...+X_{j-1})+i+j} \\ &\quad= X_j(f)\alpha(X_0, ..., X_i, ..., \hat{X}_j, ..., X_n)(-1)^{X_j(f+X_i)+1+(X_j+f)\alpha+f(X_0+...+X_{i-1})+X_j(X_0+...+\hat{X}_i+...+X_{j-1})+j} \\ &\quad+ f\alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_n)(-1)^{f\alpha+(f+X_i)(X_0+...+X_{i-1})+X_j(X_0+...+\hat{X}_i+...+X_{j-1})+i+j};\end{aligned}$$

the first term of each sum cancels with each other and we obtain linearity. Then it suffices to prove the formula for commuting vector fields, so write $\alpha = \sum f_I d\xi^I$ in local coordinates $\{\xi^a\}_a$ and notice that $d\alpha = \sum (-1)^{p(f_I)p(\xi^a)} \frac{\partial f_I}{\partial \xi^a} d\xi^a d\xi^I$, so

$$\begin{aligned}d\alpha(\frac{\partial}{\partial \xi^{i_0}}, ..., \frac{\partial}{\partial \xi^{i_n}}) &= \sum (-1)^{f_I \xi^a} \frac{\partial f_I}{\partial \xi^a} d\xi^a d\xi^I (\frac{\partial}{\partial \xi^{i_0}}, ..., \frac{\partial}{\partial \xi^{i_n}}) \\ &= \sum \frac{\partial f_I}{\partial \xi^a} d\xi^a(\frac{\partial}{\partial \xi^{i_j}}) d\xi^I(\frac{\partial}{\partial \xi^{i_0}}, ..., \widehat{\frac{\partial}{\partial \xi^{i_j}}}, ..., \frac{\partial}{\partial \xi^{i_n}})(-1)^{f_I \xi^a}(-1)^{\xi^I i_j}(-1)^{i_j(i_0+...+i_{j-1})+j} \\ &= \sum \frac{\partial}{\partial \xi^{i_j}} \left(f_I d\xi^I (\frac{\partial}{\partial \xi^{i_0}}, ..., \widehat{\frac{\partial}{\partial \xi^{i_j}}}, ..., \frac{\partial}{\partial \xi^{i_n}}) \right) (-1)^{(f_I+\xi^I)i_j}(-1)^{i_j(i_0+...+i_{j-1})+j} \\ &= \sum \frac{\partial}{\partial \xi^{i_j}} (\alpha(\frac{\partial}{\partial \xi^{i_0}}, ..., \widehat{\frac{\partial}{\partial \xi^{i_j}}}, ..., \frac{\partial}{\partial \xi^{i_n}}))(-1)^{i_j(\alpha+i_0+...+i_{j-1})+j},\end{aligned}$$

which is precisely what we wanted to show. □

Definition 2.23. *The* **Euler vector field** *on a graded manifold is the vector field* $E : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ *defined by* $E(f) = w(f)f$ *on homogeneous elements. If* $\{x^i\}_i$ *are local coordinates,*

$$E = \sum_i w(x^i)x^i \frac{\partial}{\partial x^i}.$$

***Lemma* 2.24.** Let $X \in Der\, C^\infty(\mathcal{M})$, $\alpha \in \Omega(\mathcal{M})$ be homogeneous. Then, $[E, X] = w(X)X$ and $\mathcal{L}_E\alpha = w(\alpha)\alpha$.

Proof.
We can directly compute, for $f \in C^\infty(\mathcal{M})$,

$$\begin{aligned}[E, X](f) &= E(X(f)) - X(E(f)) = w(X(f))X(f) - X(w(f)f)\\ &= (w(X) + w(f))X(f) - w(f)X(f) = w(X)X(f)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_E f &= E(f) = w(f)f\\ \mathcal{L}_E(df) &= d(\iota_E df) = d(E(f)) = d(w(f)f) = w(f)df.\end{aligned}$$

Since $\Omega(\mathcal{M})$ is locally generated as an algebra by functions $f \in C^\infty(\mathcal{U})$ and exact forms $df \in \Omega^1(\mathcal{U})$, the above suffices to conclude the Lemma by Leibniz's rule for $\mathcal{L}_E$. $\square$

It is also possible to develop an integration theory for graded manifolds, and it involves some non-trivial considerations regarding the *super* analog of the determinant, which is called the **Berezinian**. This can be found in the context of supermanifolds, as well as a whole theory of principal bundles, connections and parallel transport, in [29] or [13].

2.4. Morphisms of Graded Manifolds

In this section we present some heuristic arguments aiming to study a structure of graded manifold on the set $Mor(\mathcal{M}, \mathcal{N})$ of **morphisms** between two graded manifolds. A proper understanding of $Mor(\mathcal{M}, \mathcal{N})$ is desirable because this is the space of fields of σ-models based on graded manifolds, such as the AKSZ formalism. However, morphisms of graded manifolds are a bit more hard to understand than what it might seem at first sight, as shown in Example 2.17. One of the difficulties is that there is no clear notion of ***points***, since the points of the underlying topological space do not have a clear relation with the whole sheaf of functions. The most appropriate way to deal with this problem is to borrow the notion of ***functor of points*** from algebraic geometry.

Definition 2.25. *Given two graded manifolds* $\mathcal{Z}$, $\mathcal{M}$, *a* $\mathcal{Z}$**-point** *of* $\mathcal{M}$ *is a morphism of graded manifolds* $\varphi : \mathcal{Z} \to \mathcal{M}$.

The idea behind Definition 2.25 is that $\mathcal{M}$ is determined by the assignment

$$\mathcal{Z} \mapsto \{\mathcal{Z}\text{-points of } \mathcal{M}\},$$

as we will see in some basic examples below. This is one of the most important results from category theory and is called Yoneda's Lemma [35]. However, we do not wish to discuss this in detail here. We just intend to introduce this language to have an intuitive picture of the sort of objects that we can call *points* because it will allow us to define a structure of graded manifold on complicated objects.

Given two graded manifolds $\mathcal{M}, \mathcal{N}$, write $Mor(\mathcal{M}, \mathcal{N})$ for the set of morphisms of graded manifolds between $\mathcal{M}$ and $\mathcal{N}$. We wish to consider this object as a graded manifold itself, so let us work backwards and write $\underline{Mor}(\mathcal{M}, \mathcal{N})$ for its graded manifold structure. The least we can require is that, for any other graded manifold $\mathcal{Z}$,

$$Mor(\mathcal{Z}, \underline{Mor}(\mathcal{M}, \mathcal{N})) = Mor(\mathcal{Z} \times \mathcal{M}, \mathcal{N}). \tag{2.4}$$

Keeping in mind the idea that $\mathcal{Z}$-points determine graded manifolds, Equation (2.4) can be used in some cases to decide what the structure of $\underline{Mor}(\mathcal{M}, \mathcal{N})$ must be. Of course, this procedure requires a more careful formalization which goes beyond the scope of this work, but we can treat here some simple cases. An approach to these issues through category theory has been proved fruitful to define graded manifolds of infinite dimension in [44], while a different approach based on considering graded manifolds as fiber products is used in [29].

Example 2.26 (Points in Ordinary Manifolds). If M is an ordinary manifold and $Spec\,\mathbb{R} = (\{*\}, \mathbb{R})$ is a singleton considered as a manifold, then there is a bijection between $Mor(Spec\,\mathbb{R}, M)$ and the set M itself: it is an easy consequence of Hadamard's Lemma that any morphism of algebras $C^\infty(M) \to \mathbb{R}$ is an evaluation at some point of M. In principle this is just a bijection of sets, but the adjunction formula tells us that the natural manifold structure $\underline{Mor}(Spec\,\mathbb{R}, M)$ on $Mor(Spec\,\mathbb{R}, M)$ is such that, for any other manifold Z,

$$Mor(Z, \underline{Mor}(Spec\,\mathbb{R}, M)) = Mor(Z \times Spec\,\mathbb{R}, M) = Mor(Z, M).$$

This identity tells us that Z-points of M are in bijection with Z-points of $\underline{Mor}(Spec\,\mathbb{R}, M)$ and thus we can identify them not just as sets, but also as manifolds. Although this example might seem vacuous, the point is that, although $Mor(Spec\,\mathbb{R}, M)$ and M are equal as sets, we still need to know the fact that they have the same Z-points in order to identify them as manifolds. This appreciation is crucial in the context of graded manifolds.

Example 2.27 (Points in Graded Manifolds). Let $\mathcal{M}$ be a graded manifold and let V be a graded vector space with the same graded dimension as $\mathcal{M}$. Let $\mathcal{V} = (\{*\}, S^*V^*)$ be the corresponding graded manifold, then points in $\mathcal{M}$ can be though of as elements $P \in Mor(\mathcal{V}, \mathcal{M})$ in the sense that for $F, G \in C^\infty(\mathcal{M})$ we have $F \neq G \Leftrightarrow \exists P \in Mor(\mathcal{V}, \mathcal{M}) : P^*F \neq P^*G$. To see this note that each $P \in Mor(\mathcal{V}, \mathcal{M})$ is given by a point in the underlying topological space, $p \in M$, and compatible (with the restrictions of the sheaf) morphisms of algebras $C^\infty(\mathcal{U}) \to S^*V^*$ for each $\mathcal{U} \subset \mathcal{M}$ with $p \in U$. Thus we can assume that we are in a coordinate neighborhood around p, and choose P given by

$$\begin{aligned} P^* : C^\infty(U) \otimes S^*V^* &\to S^*V^* \\ f \otimes v &\mapsto f(p) \otimes v. \end{aligned}$$

If $F \neq G \in C^\infty(\mathcal{M})$, there exists some open neighborhood $\mathcal{U} \subset \mathcal{M}$ in which F, G can be written in local coordinates with some different coefficient $f v_1^{\alpha_1} ... v_n^{\alpha_n} \neq g v_1^{\alpha_1} ... v_n^{\alpha_n}$. Taking

$p \in M$ such that $f(p) \neq g(p)$ and defining P as above will detect the difference between F and G. Note that there are other elements in $Mor(\mathcal{V}, \mathcal{M})$ acting non-trivially on the S^*V^* component, so this object is already quite complicated as a set.

Example 2.28 (Graded Paths). The following is an example in which the structure of graded manifold on $Mor(\mathcal{M}, \mathcal{N})$ can be explicitly computed. If $\mathcal{M}$ is a graded manifold, let us study $Mor(\mathbb{R}[k,\epsilon], \mathcal{M})$ for any $(k,\epsilon) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. An element in $Mor(\mathbb{R}[k,\epsilon], \mathcal{M})$ can be thought of as a **graded path** on $\mathcal{M}$ because it pulls back functions on $\mathcal{M}$ to functions defined over a single variable t with $w(t) = k$, $p(t) = \epsilon$. Because these must preserve the grading, for coordinates $\{x^a\}_a$ on $\mathcal{M}$ an element of $Mor(\mathbb{R}[k,\epsilon], \mathcal{M})$ can only take the form

$$\begin{aligned} C^\infty(\mathcal{M}) &\to S^*(\mathbb{R}[k,\epsilon]^*) \\ x_a &\mapsto \lambda_n t^n. \end{aligned}$$

for $w(x_a) = nk$ and $p(x_a) = p(\epsilon)$. On the contrary, for $\mathcal{Z}$ any other graded manifold, the $\mathcal{Z}$-points of $\underline{Mor}(\mathbb{R}[k,\epsilon], \mathcal{M})$ are, by Equation 2.4, morphisms that locally look like

$$\begin{aligned} \varphi : C^\infty(\mathcal{M}) &\to C^\infty(\mathcal{Z}) \otimes S^*(\mathbb{R}[k,\epsilon]^*) \\ x_a &\mapsto \sum_{n\geq 0} \frac{1}{n!} \varphi_{a,n} t^n. \end{aligned}$$

with $\varphi_{a,n} \in C^\infty(\mathcal{Z})$ such that $w(x_a) = w(\varphi_{a,n}) + kn$ and $p(x_a) = p(\varphi_{a,n}) + n\epsilon$. Notice that, if $\epsilon = 1$, then $t^2 = 0$ and the above sum only has two terms. We should see the functions $\varphi_{a,n}$ as coordinates on the graded manifold $\underline{Mor}(\mathbb{R}[k,\epsilon], \mathcal{M})$; in particular, those $\varphi_{a,n}$ with $w(\varphi_{a,n}) = p(\varphi_{a,n}) = 0$ recover the original points of the set $Mor(\mathbb{R}[k,\epsilon], \mathcal{M})$. Under a coordinate change on $\mathcal{M}$ of the form $x_a = f(y)$ for some other coordinates $\{y_a\}_a$ on $\mathcal{M}$ with $\varphi(y_a) = \sum \frac{1}{n!}\psi_{a,n} t^n$, by equating $f(\sum \frac{1}{n!}\psi_{a,n} t^n) = \sum \frac{1}{n!}\varphi_{a,n} t^n$ we see that the coordinates $\varphi_{a,n}$ must transform as

$$\begin{aligned} \varphi_{a,0} &= f(\psi_0), \\ \varphi_{a,1} &= \psi_{\alpha,1} \frac{\partial f}{\partial y_\alpha}(\psi_0), \\ \varphi_{a,2} &= \psi_{\alpha,1}\psi_{\beta,1} \frac{\partial^2 f}{\partial y_\beta \partial y_\alpha}(\psi_0) + \psi_{\alpha,2} \frac{\partial f}{\partial y_\alpha}(\psi_0), \ \ldots \end{aligned}$$

We recognize these as the coordinate changes of jets on $\mathcal{M}$. If $\epsilon = 1$, only the first two terms are nonzero, and the second one is assigned parity 1, so we see $\underline{Mor}(\mathbb{R}[k,1], \mathcal{M}) = T[k,1]\mathcal{M}$, as in Example 2.14. On the other hand, when $k = 0$ we obtain an infinite number of coordinates and we can think of $\underline{Mor}(\mathbb{R}[k,0], \mathcal{M})$ as the inverse limit as $n \to \infty$ of the manifolds of n-jets on $\mathcal{M}$, where an n-jet on the direction of x_a is assigned weight $w(x_a) - nk$ and parity $p(x_a) - n\epsilon$. This construction is analogous to the construction of the tangent (or jet) space of an ordinary manifold as classes of curves on the manifold.

What we see in Example 2.28 is that that the graded manifold $\underline{Mor}(\mathcal{M}, \mathcal{N})$ has $Mor(\mathcal{M}, \mathcal{N})$ as underlying topological set, but it also has coordinates of nonzero degree representing morphisms that depend on parameters of these degrees. Such morphisms can appear in a relatively natural way, as it is the case of the flow of a non-zero degree vector field, which we proceed to define.

Definition 2.29. *Let $X \in Der\, C^\infty(\mathcal{M})$ be a vector field with $w(X) = -k$ and $p(X) = \epsilon$. A* **flow** *for X is a morphism of graded manifolds $\varphi_X : \mathcal{M} \times \mathbb{R}[k, \epsilon] \to \mathcal{M}$ satisfying*

1. $0^*(\varphi_X^* f) = f \quad \forall f \in C^\infty(\mathcal{M})$,
2. $\frac{\partial}{\partial t}\varphi_X^* f = \varphi_X^* X(f) \quad \forall f \in C^\infty(\mathcal{M})$.

Here t is the variable on $\mathbb{R}[k, \epsilon]$ and $0^* : C^\infty(M) \otimes S^*(\mathbb{R}[k, \epsilon]^*) \to C^\infty(M)$ is the morphism of algebras that sets $t = 0$.

Notice that the time parameter must be assigned weight and parity so that the two sides of the equality in 2 have the same degree because φ_X respects the grading. Thus, using Equation 2.4, the flow of X can be seen either as an $\mathcal{M}$-point on the graded manifold $\underline{Mor}(\mathbb{R}[k, \epsilon], \mathcal{M})$ studied in Example 2.28 or, more naturally, as a graded path on the graded manifold $\underline{Mor}(\mathcal{M}, \mathcal{M})$.

Iterating the relations defining the flow of X we see that

$$0^* \frac{\partial^k}{\partial t^k}\varphi_X^* f = 0^* \varphi_X^* X^k(f) = X^k(f) \quad , \forall f \in C^\infty(\mathcal{M}).$$

Thus, if we write $\varphi_X^* f = \sum_n f_n t^n$, we obtain:

1. If $\epsilon = 0$, then φ_X exists (and is unique) always; it is given by $\varphi_X^* f = \sum_n \frac{X^n(f)}{n!} t^n$. There is no need for studying convergence of this series because our functions are formal power series
2. If $\epsilon = 1$, then $t^2 = 0$ and so φ_X exists (and is unique) if and only if $X^2 = 0$, in which case $\varphi_X^* f = f - tX(f)$.

Odd vector fields squaring to zero are the basis of Q-manifolds, which we study in detail in Section 3.3 and thereafter. These are central objects in graded geometry which unify very different geometric constructions.

CHAPTER 3

Classes of Graded Manifolds

In this chapter we show how the language of graded manifolds can be used to treat many different geometric objects from a unifying point of view. In Section 3.1 we define **derived brackets**, which are a way of obtaining new interesting algebraic operations on a differential Lie algebra which will be extensively used throughout this work. In particular, the structure of L_∞**-algebras** presented in Section 3.2 can be understood in terms of (higher) derived brackets. In Section 3.3 we define a class of graded manifolds known as ***Q-manifolds***, which are the non-linear version of L_∞-algebras and unify objects such as Lie algebroids, Poisson manifolds or Courant algebroids and their cohomology theories. In Section 3.4 we show how to extend the basics of **symplectic** geometry to the graded setting. Finally, in Section 3.5, we study **non-negatively graded** manifolds and prove Vaintrob's Theorem on the characterization of Lie algebroids as a particular class of graded manifolds.

3.1. Derived Brackets

In this section we introduce the concept of a **derived bracket**. This is an algebraic operation that can be defined in any differential Lie algebra and which satisfies interesting properties; mainly, it verifies the Jacobi identity. As we will see throughout this work, derived brackets appear naturally in different contexts and are the underlying idea behind complicated geometric structures, such as Courant algebroids. We also show here that the structure itself of a differential Lie algebra can always be encoded in terms of a derived bracket, which will motivate the study of L_∞-algebras in Section 3.2.

Definition 3.1. *For $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, a* **differential Lie superalgebra of parity** *ϵ (or* **even/odd differential Lie superalgebra***) is a Lie superalgebra of parity ϵ as in Definition 2.5 with an odd linear operation $d : \mathfrak{g} \to \mathfrak{g}$ satisfying:*

1. $d^2 u = 0$.
2. $d[u, v] = [du, v] + (-1)^{p(u)+\epsilon}[u, dv]$.

The **derived bracket** *on a differential Lie superalgebra is the operation*

$$[u, v]_d := (-1)^{p(u)}[du, v]$$

and the **skew-symmetrized derived bracket** *is the operation*

$$[\![u,v]\!]_d := \frac{1}{2}\left([u,v]_d - (-1)^{(p(u)+\epsilon+1)(p(v)+\epsilon+1)}[u,v]_d\right).$$

***Proposition* 3.2.** The following properties are satisfied.

1. $[u,[v,w]_d]_d = [[u,v]_d,w]_d + (-1)^{(p(u)+\epsilon+1)(p(v)+\epsilon+1)}[v,[u,w]_d]_d$.
2. $d([u,v]_d) = [du,v]_d + (-1)^{p(u)+\epsilon+1}[u,dv]_d$.
3. $[u,v]_d = [\![u,v]\!]_d + \frac{1}{2}(-1)^{p(u)}d[u,v]$.

Proof.
All these properties follow from a direct computation.

$$\begin{aligned}
[u,[v,w]_d]_d &= (-1)^{p(u)+p(v)}[du,[dv,w]] \\
&= (-1)^{p(u)+p(v)}\left([[du,dv],w] + (-1)^{(p(u)+\epsilon+1)(p(v)+\epsilon+1)}[dv,[du,w]]\right) \\
&= (-1)^{p(v)+\epsilon+1}[d[du,v],w] + (-1)^{(p(u)+1)(p(v)+1)}[v,[u,w]_d]_d) \\
&= [[u,v]_d,w]_d + (-1)^{(p(u)+\epsilon+1)(p(v)+\epsilon+1)}[v,[u,w]_d]_d. \\
d([u,v]_d) &= (-1)^{p(u)}d[du,v] = (-1)^{\epsilon+1}[du,dv] = (-1)^{p(u)+\epsilon+1}[u,v]_d \\
&= [du,v]_d + (-1)^{p(u)+\epsilon+1}[u,v]_d. \\
2[\![u,v]\!]_d + (-1)^{p(u)}d[u,v] &= (-1)^{p(u)}[du,v] + (-1)^{(p(v)+1+\epsilon)(p(u)+\epsilon)+\epsilon}[dv,u] + (-1)^{p(u)}d[u,v] \\
&= (-1)^{p(u)}[du,v] + (-1)^{\epsilon+1}[u,dv] + (-1)^{p(u)}[du,v] + (-1)^{\epsilon}[u,dv] \\
&= 2[u,v]_d.
\end{aligned}$$

□

The way to read Proposition 3.2 is that a differential Lie superalgebra $(\mathfrak{g},[\cdot,\cdot],d)$ of parity ϵ induces a structure $(\mathfrak{g},[\cdot,\cdot]_d,d)$ of differential ***Loday*** superalgebra on $\mathfrak{g}$ of parity $1+\epsilon$ whose bilinear bracket $[\cdot,\cdot]_d$ fails to satisfy skew-symmetry by a d-coboundary. Alternatively, one can see the skew-symmetric bracket $[\![\cdot,\cdot]\!]_d$ as the fundamental object failing to satisfy the Jacobi identity by a homotopy term.

***Corollary* 3.3.** Let $(\mathfrak{g},[\cdot,\cdot],d)$ be a differential Lie superalgebra of parity ϵ and let $\mathfrak{h}\subset\mathfrak{g}$ be an abelian subalgebra closed under $[\cdot,\cdot]_d$. Then $(\mathfrak{h},[\cdot,\cdot]_d)$ is a Lie superalgebra of parity $1+\epsilon$. If, moreover, $\mathfrak{h}$ is closed under d, then $(\mathfrak{h},[\cdot,\cdot]_d,d)$ is a differential Lie superalgebra of parity $1+\epsilon$.

Example* 3.4 *(Poisson Bracket as a Derived Bracket). Given an ordinary manifold M, consider the odd Lie algebra $\Gamma(\Lambda^* TM)$ of its multivector fields with the Schouten bracket

$$[X_1\wedge\ldots\wedge X_p, Y_1\wedge\ldots\wedge Y_q] = \sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_p\wedge Y_1\wedge\ldots\wedge\hat{Y}_j\wedge\ldots\wedge Y_q$$

(see for example [38]). A **Poisson bivector** is a $\pi\in\Gamma(\Lambda^2 TM)$ such that $[\pi,\pi]=0$; in this case we see that $d_\pi := ad_\pi = [\pi,\cdot]$ is a differential on $\Gamma(\Lambda^* TM)$. Thus, it gives rise to a derived bracket which, restricted to the abelian subalgebra $C^\infty(M)$, induces an ordinary Lie algebra structure $\{\cdot,\cdot\}$ on $C^\infty(M)$ by

$$\{f,g\} = [[\pi, f], g].$$

This is precisely the Poisson bracket that the bivector π is well-known to induce on $C^\infty(M)$.

Many examples of differentials (and thus, of derived brackets) on a Lie superalgebra $\mathfrak{g}$ of parity ϵ are constructed, as in Example 3.4, by choosing an element $\Delta \in \mathfrak{g}$ of parity $1+\epsilon$ such that $[\Delta,\Delta] = 0$ and considering $d = ad_\Delta$. The rest of this chapter will provide many examples of these.

Example 3.5 (Every Lie Bracket is a Derived Bracket). Let $(V, [\cdot,\cdot]_V)$ be an ordinary Lie algebra, we claim that its Lie bracket can always be seen as the restriction of the derived bracket of a differential Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot], d)$ to an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$, as in Corollary 3.3. Indeed, consider the graded manifold $V[1] = (\{*\}, S^*(V[1]^*))$ and take $\mathfrak{g} = Der\, C^\infty(V[1])$ with the commutator as Lie bracket. The **Chevalley-Eilenberg differential** is $d_{CE} \in \mathfrak{g}_1$ (the degree 1 part of $\mathfrak{g}$) defined on $\Pi\xi \in V[1]^*$ as

$$d_{CE}(\Pi\xi) \in S^2(V[1]^*), \qquad d_{CE}(\Pi\xi)(\Pi u, \Pi v) = \xi([u,v]_V)$$

(Π denotes the parity shift $V \to V[1]$) and extended by Leibniz. If $\{e_i\}_{i=1}^n$ is a basis of V such that $[e_i, e_j]_V = c_{i,j}^k e_k$ with dual basis $\{\xi^i\}_{i=1}^n \in V^*$, we can write d_{CE} as (see Section 2.1 for sign conventions, we also write in what follows $\xi^i = \Pi\xi^i \in V[1]^*$ and $e_i = \Pi e_i \in V[1]$ to simplify notation)

$$d_{CE} = -\frac{1}{2}\sum_{i,j,k=1}^n c_{i,j}^k \xi^i \xi^j \iota_{e_k} = \sum_{k=1}^n \xi^k([\cdot,\cdot]_V)\iota_{e_k},$$

where $\iota_{e_k}(\alpha) = \alpha(\cdot, \dots, \cdot, e_k)$, as in Remark 2.4. Notice that

$$[d_{CE}, d_{CE}](\xi) = d_{CE}^2(\xi) = \sum_{k=1}^n \xi^k([\cdot,\cdot]_V)\cdot \iota_{e_k}(\xi([\cdot,\cdot]_V) = \sum_{k=1}^n \xi^k([\cdot,\cdot]_V)\cdot \xi([\cdot, e_k]_V),$$

which is an element in $S^3(V[1]^*)$ acting on $u, v, w \in V[1]$ as

$$\begin{aligned} d_{CE}^2(\xi)(u,v,w) &= \sum_{k=1}^n \big(\xi^k([u,v]_V)\cdot\xi([w,e_k]_V) - \xi^k([u,w]_V)\cdot\xi([v,e_k]_V) + \xi^k([v,w]_V)\cdot\xi([u,e_k]_V)\big) \\ &= \xi([w,[u,v]_V]_V - [v,[u,w]_V]_V + [u,[v,w]_V]_V) = 0. \end{aligned}$$

So $d_{CE} \in \mathfrak{g}_1$ satisfiees $[d_{CE}, d_{CE}] = 0$ and thus it induces a derived bracket on $\mathfrak{g}$ as $[D_1, D_2]_{d_{CE}} := (-1)^{p(D_1)}[[d_{CE}, D_1], D_2]$. In particular, $\mathfrak{g}_{-1}$ is an abelian subalgebra closed under $[\cdot,\cdot]_{d_{CE}}$, so this induces a structure of odd Lie superalgebra on $\mathfrak{g}_{-1}$. Now, as a vector space, $\mathfrak{g}_{-1} \cong V[1]$ canonically, since degree -1 derivations of $S^*(V[1]^*)$ are $\mathbb{R}$-linear maps $V[1]^* \to \mathbb{R}$ extended through Leibniz's rule and so $\mathfrak{g}_{-1} \cong V[1]^{**} \cong V[1]$ by $\iota_v \leftrightarrow v$. Thus we have an odd Lie superalgebra structure $[\cdot,\cdot]_{d_{CE}}$ on $V[1]$, which is the same as an ordinary Lie algebra structure on V; we claim that $[\cdot,\cdot]_{d_{CE}} = [\cdot,\cdot]_V$. To see this consider any

$$Q = -\sum_{i,j,k} A_{i,j}^k \xi^i \xi^j \iota_{e_k} \in \mathfrak{g}_1$$

and notice that, for $\iota_u, \iota_v \in \mathfrak{g}_{-1}$

$$-[[Q, \iota_u], \iota_v](\xi^k) = -\iota_v \iota_u Q(\xi^k) = \sum_{i,j=1}^{n} A^k_{i,j} \xi^i(u) \xi^j(v).$$

This proves that $[\cdot,\cdot]_{d_{CE}} = [\cdot,\cdot]_V$ but it also shows something more interesting: For any Q defined as above, the expression $\{u, v\}_Q := -[[Q, D_u], D_v]$ determines a skew-symmetric bracket on V and the above computations imply that this bracket satisfies the Jacobi identity if and only if $Q^2 = 0$.

Example 3.6 (Differential Lie Algebras as Derived Brackets). Considering Example 3.5, a natural question is whether we can also obtain the whole structure of a differential Lie superalgebra $(V, [\cdot,\cdot]_V, d_V)$ from a derived bracket of a different differential Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot], d_{CE})$. Indeed, this can be done, and the construction is very similar. Consider as before $\mathfrak{g} = Der\, C^\infty(V[1])$ and define now, for $\Pi\xi \in V[1]^*$, $d_{CE}(\Pi\xi) = d^0_{CE}(\Pi\xi) + d^1_{CE}(\Pi\xi) \in S^1(V[1]^*) \oplus S^2(V[1]^*)$ as

$$d^0_{CE}(\Pi\xi)(\Pi u) = \xi(du), \qquad d^1_{CE}(\Pi\xi)(\Pi u, \Pi v) = (-1)^{p(u)} \xi([u, v]).$$

As before, we write in what follows $\Pi\xi = \xi$ and $\Pi u = u$ to simplify notation. If $\{e_i\}_{i=1}^n$ is a basis of V with $de_i = c^k_i e_k$ and $(-1)^{p(e_i)}[e_i, e_j] = c^k_{i,j} e_k$, then d_{CE} can be written as

$$d_{CE} = \sum_{i,k} c^k_i \xi^i \iota_{e_k} + \frac{1}{2} \sum_{i,j,k=1}^{n} c^k_{i,j} \xi^j \xi^i \iota_{e_k}.$$

Then

$$d^2_{CE}(\xi) = d^0_{CE}(\xi^k) \cdot \iota_{e_k} d^0_{CE}(\xi) + \left(d^1_{CE}(\xi^k) \cdot \iota_{e_k} d^0_{CE}(\xi) + d^0_{CE}(\xi^k) \cdot \iota_{e_k} d^1_{CE}(\xi) \right) + d^1_{CE}(\xi^k) \cdot \iota_{e_k} d^1_{CE}(\xi),$$

which is $d^2_{CE}(\xi) = J^1 + J^2 + J^3$ for r-linear maps J^r acting on V as

$$\begin{aligned} J^1(u) &= \xi(d^2 u), \\ J^2(u, v) &= (-1)^{p(u)} \xi \left(d[u, v] - [du, v] + (-1)^{p(v)p(u)} [dv, u] \right), \\ J^3(u, v, w) &= (-1)^{p(v)} \xi \left([[u, v], w] - [u, [v, w]] + (-1)^{p(u)p(v)} [v, [u, w]] \right), \end{aligned}$$

so $d^2_{CE} = 0$ precisely when all the axioms in Definition 2.5 are satisfied. In order to recover d_V and $[\cdot,\cdot]_V$ from d_{CE} we write $D(0) \in \mathfrak{g}_{-1}$ for the degree -1 component of any $D \in \mathfrak{g}$ (the notation is chosen so that $D(0)$ equals D after setting $\xi^i = 0, \forall i$) and we recall that $\mathfrak{g}_{-1} \cong V[1]$; it is then easy to see that

$$\Pi d_V(u) = [Q, D_u](0), \qquad \Pi[u, v]_V = (-1)^{p(u)+1} [[Q, D_u], D_v] = (-1)^{p(u)+1} [[Q, D_u], D_v](0).$$

3.2. L_∞-algebras

In this section we present L_∞-algebras as a natural generalization of differential Lie algebras. They can be understood as Lie algebras which fail to satisfy the Jacobi identity by a homotopy term measured by a 3-bracket. This 3-bracket fails to satisfy a higher analog of the Jacobi identity by a further homotopy term measured by a 4-bracket, and this idea continues indefinitely. We will present a non-trivial example and discuss some of their applications.

Definition 3.7. *An L_∞-**(super)algebra** or **strongly homotopy (super)algebra** is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V with an odd vector field $d_{CE} \in Der\, C^\infty(V[1])$ such that $d_{CE}^2 = 0$. It is a **Lie n-algebra** if $V = V_0 \oplus \dots \oplus V_{-n+1}$ is $\mathbb{Z}$-graded with parity equal to weight mod 2 and $w(d_{CE}) = 1$ with respect to the corresponding grading on $V[1]$. The **Chevalley-Eilenberg algebra** of an L_∞-algebra is the differential graded algebra $(C^\infty(V[1]), d_{CE})$.*

The structure of an L_∞-algebra can be described in terms of an infinite number of multilinear operations $\{\cdot, \dots, \cdot\}_m : V \otimes \dots \otimes V \to V$ satisfying certain axioms and which are constructed from d_{CE} through **higher derived brackets** [55], in a similar way as in Examples 3.5 and 3.6. Specifically, if $\{e_i\}_i$ is a basis of V with dual basis $\{\xi^i\}_i$ and we write for simplicity $\xi^i = \Pi\xi^i$ on $V[1]$,

$$d_{CE} = \sum_{k=1}^{n} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1,\dots,i_m} A^k_{i_1,\dots,i_m} \xi^{i_1} \dots \xi^{i_m} \right) \iota_{e_k},$$

then we identify each map $A^k_{i_1,\dots,i_m} \xi^{i_1} \dots \xi^{i_m} \in S^m(V[1]^*)$ with a super skew-symmetric multilinear map of parity $m \mod 2$ via the décalage isomorphism (Remark 2.7):

$$\{e_1, \dots, e_m\} := (-1)^\epsilon \frac{1}{m!} A^k_{i_m,\dots,i_1} e_k,$$

with $\epsilon = \sum_{i=1}^m (n-i)p(e_i)$. Alternatively, using the identification $Der_{-1}\, C^\infty(V[1]) \cong V[1]$, we can define the bracket $\{\cdot, \dots, \cdot\}_m$ as

$$\iota(\{v_1, \dots, v_m\}) := (-1)^\epsilon [[Q, \iota(v_1)], \iota(v_2)], \dots, \iota(v_m)](0).$$

In the case of a Lie n-algebra, the m-bracket has weight $m-2$. The 0-bracket is a distinguished element $Q(0) = \Phi \in V$, called the **curvature** of the L_∞-algebra. The condition that $d_{CE}^2 = 0$ is then equivalent to the vanishing of the following multilinear maps on V which appear as the coefficients of d_{CE}^2:

$$J^r(v_1, \dots, v_m) = \sum_{p+q=r} \sum_{\sigma \in S_{p,q}} \{\{v_{\sigma(1)}, \dots, v_{\sigma(p)}\}, v_{\sigma(p+1)}, \dots, v_{\sigma(q)}\}(-1)^{pq}\, sgn(\sigma)\, (-1)^\gamma,$$

where $(-1)^\gamma$ is the Koszul sign and $S_{p,q}$ are the permutations of $\{1, \dots, r\}$ with $\sigma(1) < \dots < \sigma(p)$ and $(p+1) < \dots < \sigma(q)$ (compare with Equation 2.3). The maps J^r are called **Jacobiators**, they represent higher Jacobi identities *up to homotopy*, meaning that the failure of each bracket to satisfy a Jacobi identity is measured by the next bracket. The first relations are

$$\begin{aligned}
0 &= \{\Phi\}, \\
0 &= \{\{u\}\} + \{\Phi, u\}, \\
0 &= \{\{u, v\}\} - \{\{u\}, v\} + (-1)^{uv}\{\{v\}, u\} + \{\Phi, u, v\} \\
0 &= \{\{u, v, w\}\} - \{\{u, v\}, w\} + (-1)^{vw}\{\{u, w\}, v\} - (-1)^{u(v+w)}\{\{v, w\}, u\} \\
&\quad - \{\{u\}, v, w\} + (-1)^{uv}\{\{v\}, u, w\} - (-1)^{w(u+v)}\{\{w\}, u, v\} + \{\Phi, u, v, w\}
\end{aligned}$$

In particular, when $\Phi = 0$ and all m-brackets vanish for $m \geq 3$, we obtain a differential Lie superalgebra where $\{\cdot,\cdot\}$ is the Lie bracket and $\{\cdot\}$ is the differential. The converse construction is the following: given a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V with a family of super skew-symmetric m-brackets of parity $m \bmod 2$ and satisfying $J^r = 0$ for all $r \geq 0$, we define the Chevalley-Eilenbeg differential of V as the vector field on $V[1]$ acting on $\xi \in V[1]^*$ as

$$d_{CE}(\xi) = \sum_{m=0}^{\infty} \xi(\{\cdot, ..., \cdot\}).$$

The notation here has to be interpreted through the décalage isomorphism in Remark 2.7, which identifies $\xi(\{\cdot, ..., \cdot\}) \in (\Lambda^m V^*)[-m+2]$ with an element in $S^m(V[1]^*)$.

Example* 3.8 *(String Lie 2-algebra). Let $\mathfrak{g}$ be a quadratic Lie algebra. This is an ordinary Lie algebra with an additional nondegenerate bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that is invariant under the adjoint action. In other words, $\langle [u_1, u_2], u_3 \rangle + \langle u_2, [u_1, u_3] \rangle = 0$. The canonical example is the Killing form on the Lie algebra of a compact simple Lie group. Then we can define $\mu \in \Lambda^3 \mathfrak{g}^*$ as $\mu(u_1, u_2, u_3) = \langle [u_1, u_2], u_3 \rangle$ and we claim that $d_{CE}\mu = 0$, where d_{CE} is the Chevalley-Eilenberg differential from Example 3.5. Indeed, we will show in Example 3.17 below that, under the usual identification of $\Lambda^* \mathfrak{g}^*$ with left-invariant differential forms on a Lie group G, d_{CE} coincides with the exterior derivative, so we can compute

$$d_{CE}\mu(u_1, u_2, u_3, u_4) = -\mu([u_1, u_2], u_3, u_4) + \mu([u_1, u_3], u_2, u_4) - \mu([u_2, u_3], u_1, u_4) + ... = 0,$$

where we have used the Jacobi identity in the last step. Consider then $V = \mathfrak{g} \oplus \mathbb{R}[1]$ and denote by b a generator of $\mathbb{R}[1]$ with dual element $\beta \in \mathbb{R}[1]^*$. The Lie bracket of $\mathfrak{g}$ can be extended to V by letting b belong to its center; here we denote it by $\{\cdot, \cdot\} : V \times V \to V$. We also construct a super skew-symmetric 3-bracket $\{\cdot, \cdot, \cdot\} : V \times V \times V \to V$ by letting b belong to its center and defining, for $u_1, u_2, u_3 \in \mathfrak{g}$,

$$\{u_1, u_2, u_3\} = \mu(u_1, u_2, u_3)b.$$

We claim that these brackets endow V with the structure of an L_∞-algebra. As we have seen, we only need to check that, for $d_{CE} \in Der\, S^*(V[1])^*$ defined by $Q(\xi) = \xi(\{\cdot, \cdot\}) + \xi(\{\cdot, \cdot, \cdot\})$ on $\xi \in V[1]^*$, we have $Q^2 = 0$. Since the image of the 3-bracket belongs to $span\{b\}$, for $\xi \in \mathfrak{g}[1]^*$ Q acts as the Chevalley-Eilenberg differential and we already now that $Q^2(\xi) = 0$. For $\beta \in \mathbb{R}[2]^*$, we see that $Q(\beta) = \beta(\{\cdot, \cdot, \cdot\}) = \mu$ and so $Q^2(\beta) = d_{CE}\mu = 0$ by the preceeding remarks. By linearity and Leibniz's rule we can conclude that $Q^2 = 0$ and that V is an L_∞-algebra, which is called the **string Lie 2-algebra** of $\mathfrak{g}$.

***Remark* 3.9.** The above construction works in much more generality. Given an L_∞-algebra V with null n-brackets for $n \geq k$ and $\mu \in S^k(V[1])^*$ with $d_{CE}(\mu) = 0$ we can construct a new L_∞-algebra as $V_\mu = V \oplus span\,\{b\}$, where $p(b) = p(\mu)$. This is done by defining a k-bracket $\{v_1, ..., v_k\} = \mu(v_1, ..., v_k)b$ for $v_1, ..., v_k \in V$ and extending the brackets of V to V_μ by letting b belong to all the centers, including the k-bracket's one. This is called a **central extension** of V by the cocycle μ.

Definition 3.10. *A **Maurer-Cartan element** on a $\mathbb{Z}$-graded L_∞-algebra V is an element $a \in V_1$ such that $\sum_{k=0}^{\infty} \frac{1}{k!}\{a, ..., a\} = 0$.*

If there is an infinite number of non-zero brackets, some notion of convergence has to be introduced to make sense of the above sum, but we will not work in that framework. One of the main reasons why L_∞-algebras were introduced is that they can be used to study **deformations** of, essentially, any object.

Example* 3.11 *(Lie Algebra-Valued Differential Forms). Consider an ordinary manifold M and a Lie group G with Lie algebra $\mathfrak{g}$. The space $\Omega(M) \otimes \mathfrak{g}$ of Lie algebra-valued differential forms on M has a natural structure of differential graded Lie algebra with differential the de Rham differential and Lie bracket given by a combination of the wedge product and the Lie bracket on $\mathfrak{g}$, as usual. The Maurer-Cartan elements of $\Omega(M) \otimes \mathfrak{g}$ are precisely Lie algebra-valued 1-forms on M representing Cartan-Ehresmann connections on the trivial principal bundle $M \times G$ whose curvature 2-form vanishes.

For V an L_∞-algebra, V-valued differential forms on M are elements of $\Omega(M) \otimes V$, which is an L_∞-algebra. The analog of Cartan-Ehresmann connections for L_∞-algebras is defined in such a way that Maurer Cartan Elements of $\Omega(M) \otimes V$ represent flat connections. The underlying reason for this is that Maurer-Cartan elements of $A \otimes V$, for A any differential graded algebra and V any L_∞-algebra, correspond to morphisms of differential graded algebras between $(C^\infty(V[1]), d_{CE})$ and A. [23]

Example* 3.12 *(Deformations of Complex Structures). Consider an holomorphic vector bundle over a complex manifold $E \to M$. Then the space $\Gamma(\Omega^{0,p}(End\ E))$ is a differential graded Lie algebra with differential given by the Dolbeault operator $\bar{\partial}$ and Lie bracket $[\alpha e_1, \beta e_2] := \alpha \wedge \beta [e_1, e_2]$ for local sections with $e_1, e_2 \in \Gamma(End\ E)$, $\alpha, \beta \in \Omega^{0,*}(M)$ and $[e_1, e_2]$ the commutator of e_1, e_2. It is a classical result (see for example [31]) that families $\{(E_t, \bar{\partial}_t)\}_{t\in I}$ of deformations of $(E, \bar{\partial})$ are in bijection with families $\{B_t\}_{t\in I} \subset \Gamma(\Omega^{0,1}(End\ E))$ such that $\bar{\partial} B_t + \frac{1}{2}[B_t, B_t] = 0$ for all $t \in I$.

In general, deformations of any structure S are usually codified in terms of a functor assigning to every space of parameters I (usually the maximal ideal of a local Artin ring) the space deformations of S over I. These functors are always equivalent (in an appropriate sense, see [17]) to the **deformation functor** of a differential graded Lie algebra $\mathfrak{g}$, which is the functor assigning to each I the space of Maurer-Cartan elements of $\mathfrak{g} \otimes I$. Moreover, gauge equivalences on the original problem can also be codified as **gauge actions** of $\mathfrak{g}$ on each set of Maurer-Cartan elements in a purely algebraic way. The advantage of introducing L_∞-algebras is that, if we see $\mathfrak{g}$ as an L_∞-algebra, then its deformation functor is preserved by quasi-isomoprhisms with other L_∞-algebras, which reveals a great flexibility on the way this functor can be presented.

L_∞-algebras (and L_∞-algebroids, which we will define in Section 3.5) are of great importance in different areas of geometry and physics. To name a few examples of their applications,

- The cornerstone of Kontsevich's proof of his Formality Theorem [33] on **deformation quantization** of Poisson manifolds is the above mentioned result on preservation of deformation functors between quasi-isomorphic L_∞-algebras. In his proof he treats L_∞-algebras as Q-manifolds, which we shall define in Section 3.3.

- L_∞-algebras can be integrated to objects called **∞-groups** [24]. These are higher analogs of groups in the sense that they can be thought of as a group in which associativity fails by a homotopy term, which fails itself to satisfy a higher associativity relation by a higher homotopy term, and this continues indefinitely in a similar way as the Jacobiators of the L_∞-algebra. These ∞-groups can be used to describe notions of equivalence between objects that are less restrictive than the idea of isomorphism, which is sometimes interesting because it can give rise to moduli spaces with a richer geometric structure.

- $BV-BRST$ **formalism** constructs an L_∞-algebroid A which models the reduced phase space of a Lagrangian field theory with gauge symmetries [9]. The quantization of such model is then performed in terms of the Feynmann path integral, which is interpreted here as the map sending each closed element in the Chevalley-Eilenberg algebra of A to its cohomology class.

- There is a well-developed theory of principal bundles, Cartan-Ehresmann connections and invariant polynomials for L_∞-algebras which can be used to define **higher Chern-Simmons theories** [45]. Field theories as diverse as topological Yang-Mills theory, the D'Auria-Fré formalism for supergravity and all AKSZ models, such as the Poisson and Courant σ-models, are particular instances of this general framework; see [27] for a complete review of these models or [1], [48], [43] for some of the original ideas.

3.3. Q-manifolds

In this section we present **Q-manifolds**, which are one of the most useful classes of graded manifolds. Essentially, a Q-manifold is a graded manifold whose sheaf of functions is a sheaf of differential graded algebras, the differential being given by a vector field Q. This provides a unifying language for many different cohomology theories. Moreover, the vector field Q induces on the Lie algebra $Der\, C^\infty(\mathcal{M})$ new algebraic operations via the derived brackets studied in Section 3.1. In particular, a Q-manifold over a point is an L_∞-algebra, so Q-manifolds can be seen as their non-linear version. Many examples, such as Poisson manifolds, Lie algebroids or Courant algebroids will appear throughout this work.

Definition 3.13. *A* **Q-manifold** *or* **dg-manifold** *is a graded manifold $\mathcal{M}$ equipped with an* **homological vector field**; *that is, an odd vector field Q with $[Q,Q]=2Q^2=0$. If the $\mathbb{Z}$-grading on $\mathcal{M}$ is non-trivial we will also require $w(Q)=1$ unless otherwise specified. The* **Q-cohomology** *of $\mathcal{M}$ is the cohomology $H^*_Q(\mathcal{M})$ of the complex defined by $C^\infty(\mathcal{M})$ and Q. A* **morphism of Q-manifolds** *or* **Q-morphism** *is a morphism of graded manifolds $\varphi:\mathcal{M}\to\mathcal{N}$ such that $Q_\mathcal{M}\circ\varphi^*=\varphi^*\circ Q_\mathcal{N}$.*

We note that the condition $Q^2=0$ can be understood as an integrability condition on Q, following the final discussion in Section 2.4.

Example 3.14 (L_∞-algebras as Q-manifolds). It follows directly from our Definition 3.7 that a Q-manifold over a point $(\{*\}, S^*V^*)$ is precisely an L_∞-algebra structure on $V[-1]$. L_∞-algebras were originally defined in terms of the multilinear brackets presented in Section 3.1, but seeing them as Q-manifolds has some advantages. For example, it is not very clear what a morphism between L_∞-algebras in the infinite-number-of-brackets definition is, because requiring that a map preserves all the brackets fails to capture the homotopy nature of these objects. It turns out that considering Q-morphisms between them is the appropriate notion at least when treating deformation problems, as it is done in [33].

Example* 3.15 *(De Rham Differential). Let M be an ordinary manifold and consider $\mathcal{M} = T[1]M = (M, \Omega(M))$. The exterior derivative of M, $d_{dR} = \sum dx^a \frac{\partial}{\partial x^a}$, is a homological vector field of weight 1 on $\mathcal{M}$. The Q-cohomology of $\mathcal{M}$ is precisely the de Rahm cohomology of M. A Q-morphism with another Q-manifold of this form is simply a morphism of the underlying manifolds because $\Omega(M)$ is locally generated as an algebra by functions and ***exact*** forms, so graded algebra homomorphisms $\Omega(M) \to \Omega(N)$ preserving d_{dR} are determined by their restriction $C^\infty(M) \to C^\infty(N)$.

More generally, note that for any graded manifold $\mathcal{M}$ with coordinates $\{x^a\}_a$ and for any $k \in \mathbb{Z}$ the tangent bundle $T[k,1]\mathcal{M}$ with coordinates $\{x^a, v^a\}_a$ in the sense of Example 2.14 is a Q-manifold with homological vector field given by

$$Q = \sum_a v^a \frac{\partial}{\partial x^a}.$$

As it has been previously mentioned (see Remark 2.20), $C^\infty(T[k,1]\mathcal{M})$ equals $\Omega(\mathcal{M})$ as a vector space but not as an algebra unless we consider the algebra structure arising from the décalage isomorphism. Thus, the above Q does not equal the exterior derivative as we defined it in Definition 2.21. Instead, it is the unique extension to an odd, degree k derivation of $C^\infty(T[k,1]\mathcal{M})$ (with the décalage grading) coinciding with the exterior derivative on $C^\infty(\mathcal{M})$. As before, Q-morphisms in this case are just morphisms of the original graded manifolds.

Example* 3.16 *(Flat Connections). Let $E \to M$ be a vector bundle, and let ∇ be a flat connection on E. This means that it is a degree 1 derivation of $\Omega(M;E) = \Omega(M) \otimes \Gamma(E)$ squaring to 0 with a local expression of the form

$$\nabla = \sum dx^i \frac{\partial}{\partial x^i} + \sum \Gamma^b_{a,i} dx^i \xi^a \frac{\partial}{\partial \xi^b}.$$

In particular, it is a homological vector field on the graded manifold $\mathcal{E} := T[1]M \oplus E[1] = (M, \Omega(M) \otimes \Lambda^*\Gamma(E^*))$.

Example* 3.17 *(Quotients of Q-manifolds). Consider a Lie group G acting on a Q-manifold $\mathcal{M}$ in such a way that Q is preserved. That is, suppose we have a group homomorphism

$$\varphi : G \to Diff(\mathcal{M})$$
$$g \mapsto \varphi_g,$$

where $Diff(\mathcal{M})$ is the group of invertible morphisms of graded manifolds $\mathcal{M} \to \mathcal{M}$, and that $\varphi_g^*(Q(f)) = Q(\varphi_g^* f)$, $\forall g \in G, \forall f \in C^\infty(\mathcal{M})$. Then, under sufficient regularity conditions, $\mathcal{M}/G = (M/G, C^\infty(\mathcal{M})/G)$ is again a Q-manifold.

Take for example $\mathcal{M} = T[1]G = (G, \Omega(G))$ with the homological vector field given by the exterior derivative, as in Example 3.15. Then G acts on $\mathcal{M}$ by pulling back the differential forms by L_g, the left multiplication by $g \in G$. The quotient consists on a single point where we consider left-invariant differential forms on G as functions. In other words, the quotient is $\mathfrak{g}[1] = (\{*\}, \Lambda^*\mathfrak{g}^*)$. Since $d_{dR} = \sum_i dx^i \frac{\partial}{\partial x^i}$ commutes with pull-back, $\mathfrak{g}[1]$ inherits a homological vector field Q. To compute it we notice that for a left-invariant one-form α and left-invariant vector fields X_0, X_1 we have $d\alpha(X_0, X_1) = -\alpha([X_0, X_1])$, which means that for $\xi \in \mathfrak{g}^*$ we have $Q(\xi) \in \Lambda^2\mathfrak{g}^*$ acting as $Q(\xi)(u, v) = -\xi([u, v])$. That is, Q is (up to a sign) the ***Chevalley-Eilenberg differential*** presented in Example 3.5.

Example 3.18 (Action Lie Algebroid). Suppose G is a Lie group acting on some ordinary manifold M. Then we have a Lie algebra homomorphism

$$\begin{aligned} \rho : \mathfrak{g} &\to \Gamma(TM) \\ u &\mapsto X_u \end{aligned}$$

sending each $u \in \mathfrak{g}$ to its fundamental vector vield X_u. The above map has dual

$$\begin{aligned} \rho^* : \Gamma(T^*M) &\to \mathfrak{g}^* \otimes C^\infty(M) \\ \alpha &\mapsto \rho^*(\alpha), \end{aligned}$$

where $\rho^*(\alpha)(u) = \alpha(X_u)$ for $u \in \mathfrak{g}$. Consider $M \times \mathfrak{g}[1] = (M, C^\infty(M) \otimes \Lambda^*\mathfrak{g}^*)$. This graded manifold is a Q-manifold if we define Q as the Chevalley-Eilenberg differential on elements of $\Lambda^*\mathfrak{g}^*$ and as $\rho^*(df)$ for elements $f \in C^\infty(M)$. That is, for $f \otimes \xi \in C^\infty(M) \otimes \mathfrak{g}^*$ we define $Q(f \otimes \xi) \in \Lambda^2\mathfrak{g}^* \otimes C^\infty(M)$ acting on $u, v \in \mathfrak{g}$ as

$$Q(f \otimes \xi)(u, v) = (Q(f)\xi + fQ(\xi))(u, v) = -X_u(f)\xi(v) + X_v(f)\xi(u) + f\xi([u, v]).$$

In coordinates, if $\{e_i\}_i$ is a basis of $\mathfrak{g}$ with fundamental vector fields $X_i = X_i^a \frac{\partial}{\partial x^a}$ and dual basis $\{\xi^i\}_i$, we have

$$Q = -\frac{1}{2}\sum_{i,j,k} c_{i,j}^k \xi^i \xi^j \frac{\partial}{\partial \xi^k} + \sum_{i,a} \xi^i X_i^a \frac{\partial}{\partial x^a} = d_{CE} + \sum_i \xi^i X_i = d_{CE} + \sum_a \rho^*(dx^a)\frac{\partial}{\partial x^a},$$

where $c_{i,j}^k = \xi^k([e_i, e_j])$ are the structure constants of $\mathfrak{g}$. Thus,

$$\begin{aligned} Q^2(f)(u, v) &= \sum_i Q(X_i(f)\xi^i)(u, v) = -X_u(X_i(f))\xi^i(v) + X_v(X_i(f))\xi^i(u) + X_i(f)\xi^i([u, v]) \\ &= -X_u(X_v(f)) + X_v(X_u(f)) + X_{[u,v]}(f) = 0, \end{aligned}$$

which implies $Q^2 = 0$ in general, since $Q^2(\xi) = d_{CE}^2(\xi) = 0$ for $\xi \in \mathfrak{g}^*$. This is an example of a **Lie algebroid**, which we will study in Section 3.5 below.

Example 3.19 (Equivariant Cohomology). Let M be an ordinary manifold endowed with the action

$$\begin{aligned} \varphi : G &\to Diff(M) \\ g &\mapsto \varphi_g \end{aligned}$$

of a compact, connected Lie group G. Consider $\mathcal{M} = \mathfrak{g}[2] \times T[1]M$, which has $C^\infty(\mathcal{M}) = S^*\mathfrak{g}^* \otimes \Omega(M)$. We think of elements $\omega = p \otimes \alpha \in C^\infty(\mathcal{M})$ as polynomials on $\mathfrak{g}$ with values on $\Omega(M)$ and say that they are ***equivariant*** if $p(Ad_g \cdot) \otimes \alpha = p \otimes \frac{d}{dt}_{|t=0} \varphi_g^* \alpha$. Equivalently, we call equivariant those elements of $C^\infty(\mathcal{M})$ that are invariant under the action

$$g \cdot (p \otimes \alpha) = p(Ad_{g^{-1}} \cdot) \otimes \frac{d}{dt}_{|t=0} \varphi_g^* \alpha.$$

We have an odd vector field Q of weight 1 on $\mathcal{M}$

$$Q(\omega)(v) = (d - \iota_{X_v})\omega(v),$$

where X_v is the fundamental vector field generated by v. Notice

$$Q^2(\omega)(v) = (d - \iota_{X_v})(d - \iota_{X_v})\omega(v) = -\mathcal{L}_{X_v}\omega(v),$$

which means precisely that $Q^2(\omega) = 0$ for equivariant forms. In particular, we see that $\mathcal{N} := (M, (S^*\mathfrak{g}^* \otimes \Omega(M)^G))$ is a Q-manifold, whose Q-cohomology is the Cartan model for the ***equivariant cohomology*** of M.

Example* 3.20 *(Q-structure on $\underline{Mor}(\mathcal{N}, \mathcal{M})$). If $(\mathcal{M}, Q_\mathcal{M})$, $(\mathcal{N}, Q_\mathcal{N})$ are Q-manifolds, then the infinite-dimensional graded manifold $\underline{Mor}(\mathcal{N}, \mathcal{M})$ has a natural structure of Q-manifold. As in Section 2.4, we cannot present this in a strictly precise way because we have not defined infinite-dimensional graded manifolds, but we can give an intuitive idea of the construction (details in [57]). If we see vector fields on $\mathcal{M}$, $\mathcal{N}$ and $\underline{Mor}(\mathcal{N}, \mathcal{M})$ as sections of their tangent bundles instead of derivations of the sheaves of functions and we identify the tangent bundle of $\underline{Mor}(\mathcal{N}, \mathcal{M})$ at $\varphi \in Mor(\mathcal{N}, \mathcal{M})$ with $\varphi^* T\mathcal{M}$, then we define

$$Q_\mathcal{N}^\mathcal{M}(\varphi) = \varphi^* Q_\mathcal{M} - \varphi_* Q_\mathcal{N},$$

where as usual the pull-back is defined as $\varphi^* Q_\mathcal{M} = Q_\mathcal{M} \circ \varphi$ and the push-forward is defined as $\varphi_* Q_\mathcal{N} = d\varphi(Q_\mathcal{N})$. Both pull-back and push-forward preserve the Lie bracket and thus $Q_\mathcal{N}^\mathcal{M}$ is a homological vector field. The importance of this construction is that, for $\mathcal{N} = T[1]\Sigma$ and $\mathcal{M}$ a symplectic NQ-manifold, the graded manifold $\underline{Mor}(\mathcal{N}, \mathcal{M})$ is the space of fields of AKSZ formalism [27], [9], as explained in Section 1.2.

As explained in Section 3.2, Maurer-Cartan elements on L_∞−algebras usually represent deformations of a structure of interest. In the language of Q-manifolds we can interpret Maurer-Cartan elements, at least formally, as zeroes of d_{CE}. Indeed, for an L_∞−algebra V, a Maurer-Cartan element $a \in V_1$ is precisely a **point** in $V[1]$ (meaning that $a \in (V[1])_0$) where the homological vector field d_{CE} vanishes, because $d_{CE}(\xi) = \sum_{m=0}^\infty \xi(\{\cdot, ..., \cdot\}_m)$ and $\xi(\{\cdot, ...\cdot\}_m) \in S^m(V[1]^*)$ can be seen as a polynomial on $(V[1])_0$ acting as $a \mapsto \frac{1}{m!}\xi(\{a, ...a\}_m)$, as in Section 2.1. For general Q-manifolds, one should expect that the *set of zeroes of* Q is an appropriate model for some kind of non-linear theory of deformations. We do not delve into this idea, which as of now has only been sketched in some places [47], [1]. In particular, notice that in Example 3.20 the set of zeroes of $Q_\mathcal{N}^\mathcal{M}$ is precisely the set of Q-morphisms $\varphi : \mathcal{N} \to \mathcal{M}$.

In all our examples Q serves on $\mathcal{M}$ a role very similar to that of the exterior derivative on an ordinary manifold M: it sends a function f of degree 0 to something which resembles a differential 1-form in that it can be evaluated at elements v of some space giving a function that represents the variation of f along the Q*-direction* given by v. We will see many other examples of Q-manifolds throughout this work, and this will still be the case, so one can think of the homological vector field Q as a tool for measuring variations along *graded directions*.

3.4. Graded Symplectic Manifolds

In this section we study **symplectic** forms in graded manifolds and the structure that they give rise to. Ordinary symplectic manifolds are used to model phase spaces in classical mechanics, while the first examples of graded symplectic manifolds appeared in the BRST formalism as an attempt to generalize some of the nice properties of these models to more complicated gauge theories by introducing *ghost fields*, which in our language are nothing but functions of non-zero degree.

Definition 3.21. *A* **symplectic form** *in a graded manifold is a* 2*-form* ω *of weight* k *and parity* ϵ *such that:*

1. *The map* $Der\, C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})[k,\epsilon]$ *sending* X *to* $\iota_X\omega$ *is an isomorphism of graded* $C^\infty(\mathcal{M})$*-modules.*
2. $d\omega = 0$.

As in the ordinary case, the existence of a symplectic form presents some restrictions on the dimension of $\mathcal{M}$. For each coordinate x^i of degree (l,η), the vector field $\frac{\partial}{\partial x^i}$ has degree $(-l,\eta)$, and the 1-form dx^i has degree (l,η). Since these generate $Der\, C^\infty(\mathcal{M})$ and $\Omega^1(\mathcal{M})$, respectively, the above isomorphism means that the dimension on each degree (l,η) must coincide with the dimension on degree $(-l+k,\eta+\epsilon)$. In particular, if ω is odd, the number of even coordinates coincides with the number of odd coordinates. If ω is even, the above isomorphism restricts to the even parts of each module, so it induces an ordinary symplectic form and thus the dimension on even coordinates must be an even integer.

Another consequence of the isomorphism in Definition 3.21 is that for every $H \in C^\infty(\mathcal{M})$ there exists some vector field X_H with $\iota_{X_H}\omega = -dH$. Notice that $w(X_H) = \omega(H) - k$ and $p(X_H) = p(H) + \epsilon$. This allows us to consider the following definition.

Definition 3.22. *A* **symplectic vector field** *is a vector field* X *such that* $\mathcal{L}_X\omega = 0$. *Equivalently,* X *is symplectic if* $d\iota_X\omega = 0$. *It is a* **Hamiltonian vector field** *if* $X = X_H$ *for some* $H \in C^\infty(\mathcal{M})$.

***Proposition* 3.23.** A symplectic form ω of weight $k \neq 0$ is always exact. A symplectic vector field X of weight l, with $k + l \neq 0$, is always Hamiltonian.

Proof.
Let E denote the Euler vector field from Definition 2.23. Then by Proposition 2.24,

$$k\omega = \mathcal{L}_E\omega = \iota_E d\omega + d\iota_E\omega = d\iota_E\omega$$

and

$$(k+l)\iota_X\omega = \mathcal{L}_E\iota_X\omega = d\iota_E\iota_X\omega + \iota_E d\iota_X\omega = d\iota_E\iota_X\omega.$$

Thus, $\omega = d(k^{-1}\iota_E\omega)$ and $\iota_X\omega = -d(-(k+l)^{-1}\iota_E\iota_X\omega)$. □

Proposition 3.24. Let X, Y be symplectic vector fields. Then $[X, Y]$ is a Hamiltonian vector field with Hamiltonian function $(-1)^{\epsilon(p(X)+p(Y))}\omega(X, Y)$.

Proof.
We use Proposition 2.22, 4 to see that

$$\begin{aligned}\iota_{[X,Y]}\omega &= [\mathcal{L}_X, \iota_Y]\omega = (\iota_X d + d\iota_X)\iota_Y\omega + (-1)^{p(X)p(Y)}\iota_Y(\iota_X d + d\iota_X)\omega \\ &= d\iota_X\iota_Y\omega = d(\omega(Y,X))(-1)^{p(Y)\epsilon+p(X)(p(Y)+\epsilon)} = -d(\omega(X,Y)(-1)^{\epsilon(p(X)+p(Y))}).\end{aligned}$$

□

Definition 3.25. *Given $f, g \in C^\infty(\mathcal{M})$ we define their* **Poisson bracket** *as the function*

$$\{f, g\} = (-1)^{\epsilon(p(f)+p(g))}\omega(X_f, X_g) = X_f(g) \in C^\infty(\mathcal{M}).$$

Proposition 3.26. The Poisson bracket satisfies the following properties.

1. $w(\{f,g\}) = w(f) + w(g) - k$ and $p(\{f,g\}) = p(f) + p(g) + \epsilon$.
2. $[X_f, X_g] = X_{\{f,g\}}$.
3. $\{f,g\} = -(-1)^{(\epsilon+p(f))(\epsilon+p(g))}\{g,f\}$.
4. $\{f,gh\} = \{f,g\}h + (-1)^{p(g)(\epsilon+p(f))}g\{f,h\}$.
5. $\{f,\{g,h\}\} = \{\{f,g\},h\} + (-1)^{(p(f)+\epsilon)(p(g)+\epsilon)}\{g,\{f,h\}\}$.
6. $\{f,g\} = 0\,\forall g \in C^\infty(\mathcal{M}) \Leftrightarrow f$ locally constant.

In particular, $(C^\infty(\mathcal{M}), \{\cdot,\cdot\})$ is a Poisson superalgebra if ω is even and a Gerstenhaber algebra if ω is odd, and part of parity ϵ of $C^\infty(\mathcal{M})$ is an ordinary Lie algebra under the Poisson bracket.

Proof.
1 is immediate, 2 follows from Proposition 3.24 and 3 follows from skew-symmetry of ω. For 4, note that $d(gh) = (dg)h + g(dh)$ implies $X_{gh} = (-1)^{\epsilon p(h)}X_g h + gX_h$, so

$$\begin{aligned}\{f,gh\} &= (-1)^{\epsilon(p(f)+p(g)+p(h))}\omega(X_f, X_{gh}) \\ &= (-1)^{\epsilon(p(f)+p(g))}\omega(X_f,X_g)h + (-1)^{\epsilon(p(f)+p(g)+p(h))+p(g)p(f)}g\omega(X_f,X_h) \\ &= \{f,g\}h + (-1)^{p(g)(\epsilon+p(f))}g\{f,h\}.\end{aligned}$$

To prove 5, we extend $0 = d\omega(X_f, X_g, X_h)$ using Proposition 2.22 6:

$$\begin{aligned}0 &= (-1)^{(f+\epsilon)\epsilon}X_f(\omega(X_g,X_h)) + (-1)^{(g+\epsilon)f+1}X_g(\omega(X_f,X_h)) + (-1)^{(h+\epsilon)(\epsilon+f+g)}X_h(\omega(X_f,X_g)) \\ &\quad - \omega([X_f,X_g],X_h) + (-1)^{(h+\epsilon)(g+\epsilon)}\omega([X_f,X_h],X_g) + (-1)^{(g+h)(f+\epsilon)+1}\omega([X_g,X_h],X_f) \\ &= (-1)^{(f+g+h+\epsilon)\epsilon}\{f,\{g,h\}\} + (-1)^{gf+\epsilon h+1}\{g,\{f,h\}\} + (-1)^{h(\epsilon+f+g)+\epsilon}\{h,\{f,g\}\} \\ &\quad + (-1)^{(f+g+h+\epsilon)\epsilon+1}\{\{f,g\},h\} + (-1)^{\epsilon f+gh}\{\{f,h\},g\} + (-1)^{f(g+h+\epsilon)+\epsilon+1}\{\{g,h\},f\} \\ &= 2(-1)^{(f+g+h+\epsilon)\epsilon}\left(\{f,\{g,h\}\} - \{\{f,g\},h\} - (-1)^{\epsilon+fg+\epsilon(f+g)}\{g,\{f,h\}\}\right),\end{aligned}$$

which gives the desired result. Finally, 6 follows from $X_f(g) = 0 \, \forall g \in C^\infty(\mathcal{M}) \Leftrightarrow -df = \iota_{X_f}\omega = 0$. $\square$

***Remark* 3.27.** Proposition 3.23 implies that when $k \neq 0$ the space $C^\infty_{k,\epsilon}(\mathcal{M})$ describes all infinitesimal symplectic transformations of $(\mathcal{M}, \omega)$, which are given by symplectic vector fields preserving weight and parity. Moreover, Proposition 3.26 implies that the Poisson bracket restricts to $C^\infty_{k,\epsilon}(\mathcal{M})$ and represents the commutator of the corresponding vector fields. This means that $(C^\infty_{k,\epsilon}(\mathcal{M}), \{\cdot,\cdot\})$ is the Lie algebra of the group of automorphisms of $\mathcal{M}$ preserving ω.

***Remark* 3.28.** If $\{\xi^i\}_i \cup \{\rho_i\}_i$ are coordinates on $\mathcal{M}$ such that $\omega = \sum_i d\rho_i d\xi^i$, then a quick computation shows that

$$\{\xi^i, \xi^j\} = \{\rho_i, \rho_j\} = 0, \qquad \{\rho_i, \xi^j\} = \delta_{ij}(-1)^{p(\rho_i)+p(\omega)(p(x^j)+1)}.$$

For the sake of completeness, we also include the following definition.

Definition 3.29. *A graded manifold $\mathcal{M}$ is a* **Poisson (graded) manifold** *if $C^\infty(\mathcal{M})$ is a sheaf of Poisson algebras. A morphism between Poisson manifolds is a* **Poisson map** *if it preserves the Poisson bracket.*

If $(\mathcal{M}, \omega)$ is a symplectic graded manifold and $i : \mathcal{N} \to \mathcal{M}$ is a submanifold, then we can pull-back $T\mathcal{M}$ and $\omega \in \Lambda^2\Gamma(T^*\mathcal{M})$ along i to obtain $i^*\omega \in \Lambda^2\Gamma(i^*T^*\mathcal{M})$, where the exterior product is as $C^\infty(\mathcal{N})$-module, see Section 2.1, and the sheaf of sections of a graded vector bundle is as defined in Example 2.13. There is also an injection of vector bundles $T\mathcal{N} \to i^*T\mathcal{M}$, so we may see $\Gamma(T\mathcal{N}) \subset \Gamma(i^*T\mathcal{M})$. If $\mathcal{N}$ is modelled on the free supercommutative algebra A and has N as underlying manifold, the localization of $i^*(T^*\mathcal{M})$ at each $p \in N$ is an A-module where $j^*\omega_{|p}$ is a non-degenerate super skew-symmetric bilinear map. Thus, we can write $\Gamma(T_p\mathcal{N})^\omega \subset \Gamma(i^*T_p\mathcal{M})$ for the orthogonal complement with respect to $i^*\omega$.

Definition 3.30. *With the notation from the previous paragraph, a submanifold $i : \mathcal{N} \to \mathcal{M}$ of a graded symplectic manifold $(\mathcal{M}, \omega)$ is*

1. ***isotropic*** *if* $\Gamma(T_p\mathcal{N}) \subset \Gamma(T_p\mathcal{N})^\omega \;\; \forall p \in N$,
2. ***coisotropic*** *if* $\Gamma(T\mathcal{N})^\omega \subset \Gamma(T\mathcal{N}) \;\; \forall p \in N$,
3. ***Lagrangian*** *if* $\Gamma(T\mathcal{N}) = \Gamma(T\mathcal{N})^\omega \;\; \forall p \in N$,
4. ***symplectic*** *if* $\Gamma(T\mathcal{N}) \cap \Gamma(T\mathcal{N})^\omega = 0 \;\; \forall p \in N$.

The combination of symplectic manifolds and Q-manifolds seems to be a powerful tool in applications to field theories. If we have a graded symplectic manifold $\mathcal{M}$ and we consider an odd **Hamiltonian** vector field Q with $Q^2 = 0$ and $w(Q) = 1$, let $S \in C^\infty(\mathcal{M})$ be its Hamiltonian function. This means that $Q(g) = \{S, g\}$, so we have $w(S) = 1 + k$, $p(S) = \epsilon + 1$, and we see that $[Q, Q] = 0$ is equivalent to $\{S, S\}$ being locally constant by Proposition 3.26. But constants have weight and parity zero, and $w(\{S, S\}) = k + 2$, $p(\{S, S\}) = \epsilon$. Thus, when $\epsilon \neq 0$ or $k \neq -2$, S must solve the ***Classical master equation*** $\{S, S\} = 0$. In any case, $\{S, \cdot\}$ is a differential on $C^\infty(\mathcal{M})$, so it gives rise to a derived bracket as in Section 3.1.

If such $Q = \{S, \cdot\}$ exists, we say that a submanifold $i : \mathcal{N} \to \mathcal{M}$ is **Q-isotropic** (resp., **Q-coisotropic**, **Q-Lagrangian**, **Q-symplectic**) if $\mathcal{N}$ is isotropic (resp., coisotropic, etc.), $\mathcal{N}$ is a Q-manifold and i is a Q-morphism. Q-Lagrangian submanifolds are called Λ-structures in [48], where their role in σ-models having $\mathcal{M}$ as target space is studied.

Example* 3.31 *(Poisson Manifolds). Let M be an ordinary manifold and consider the graded manifold $\mathcal{M} = T^*[1]M = (M, \Gamma(\Lambda^* TM))$. We can define an odd symplectic form of weight 1 which in local coordinates (x^i, p_i) can be written as $\omega = dp_i dx^i$, where x^i are the base coordinates and p_i are the fiber coordinates of T^*M. This means that $\{x^i, x^j\} = \{p_i, p_j\} = 0$ and $\{p_i, x^j\} = \delta_{ij}$. Thus, if $X, Y \in C_1^\infty(\mathcal{M}) = \Gamma(TM)$ and $f \in C_0^\infty(\mathcal{M}) = C^\infty(M)$, we obtain

$$\{X, f\} = X(f) \qquad \{\{X, Y\}, f\} = [X, Y](f).$$

This shows that the Poisson bracket in this example coincides with the Schouten bracket for multivector fields, since it is the only possible extension of the above relations making $C^\infty(\mathcal{M})$ a Gerstenhaber algebra. Recall that in Example 3.4 we said that an element $\pi \in \Gamma(\Lambda^2 TM)$ induces via derived brackets a Poisson bracket on $C^\infty(M)$ if and only if $\{\pi, \pi\} = 0$, for $\{\cdot, \cdot\}$ the Schouten bracket. With this new persepective, we can say that *a degree 2 solution to the classical master equation on $T^*[1]M$ is equivalent to a Poisson tensor on M.*

Note that isotropic (resp. coisotropic, etc.) submanifolds of $T^*[1]M$ are **not** in bijection with ordinary isotropic (resp. coisotropic, etc.) submanifolds of T^*M because morphisms of graded manifolds $i : \mathcal{N} \to \mathcal{M}$ are required to preserve the grading; we will study this in more detail in Remark 4.12. It is also interesting to notice that 1-forms α on M correspond to odd vector fields ι_α of weight -1 on $\mathcal{M}$ with ι_α being symplectic whenever α is closed and Hamiltonian whenever α is exact (compare with Proposition 3.23).

Example* 3.32 *(Cotangent Bundles are Symplectic). Example 3.31 admits the following generalization. If $\mathcal{M}$ is any graded manifold, then $T^*[k, \epsilon]\mathcal{M}$ is a symplectic graded manifold, with ω of degree k and parity ϵ. If $\{\xi^i\}_i$ are coordinates on $\mathcal{M}$ of arbitrary parities and weights, then $T^*[k, \epsilon]\mathcal{M}$ has coordinates $\{\xi^i, \rho_i\}_i$ with $deg(\rho_i) = k - deg(\xi^i)$ and $p(\rho_i) = \epsilon - deg(\xi_i)$ and we can write ω as

$$\omega = d\rho_i d\xi^i.$$

This is globally well-defined because the computations are the same as for ordinary cotangent bundles. Notice $C^\infty(T^*[k, \epsilon]\mathcal{M}) = S^*(Der\, C^\infty(M)[-k, \epsilon])$ as $C^\infty(M)$-modules but with the algebra structure coming from the décalage isomorphism (see Remarks 2.7 and 2.20), and the Poisson bracket on $T^*[k, \epsilon]\mathcal{M}$ extends the relations $\{X, f\} = X(f)$, $\{\{X, Y\}, f\} = [X, Y](f)$ through Leibniz's rule with respect to this algebra structure.

As in ordinary symplectic geometry, for $X = f_i(\xi)\frac{\partial}{\partial \xi^i}$ a vector field on $\mathcal{M}$ we can define its **Hamiltonian lift** as the vector field X^L on $T^*[k, \epsilon]\mathcal{M}$ defined by $X^L = \{\pm f_i(\xi^i)\rho_i, \cdot\}$, with the appropriate signs so that X^L coincides with X on $C^\infty(\mathcal{M}) \subset C^\infty(T^*[k, \epsilon]\mathcal{M})$.

Note $w(X^L) = w(X)$ and $p(X^L) = p(X)$. As usual, one can easily check using the Poisson bracket that $[X^L, Y^L] = [X, Y]^L$ so, in particular, if $\mathcal{M}$ has a homological vector field Q then its Hamiltonian lift Q^L is also Hamiltonian. In conclusion, for $\mathcal{M}$ a Q-manifold, the graded manifold $T^*[k,\epsilon]\mathcal{M}$ is canonically a symplectic Q-manifold. Moreover, any $H \in C^\infty(\mathcal{M})$ with $p(H) = \epsilon + 1$ and $Q(H) = 0$ defines a new homological vector field on $T^*[k,\epsilon]\mathcal{M}$ as $Q^L + \{H, \cdot\}$.

Example* 3.33 *(BRST Formalism). As a particular case of Example 3.32 and recalling Example 3.18, consider an ordinary manifold M with the action of a Lie group G. Let $E = M \times \mathfrak{g}$ with coordinates $\{x^a, \xi^i\}$ and define $\mathcal{M} = T^*[2,1]E[1]$ with coordinates $\{x^a, \xi^i, p_a, \rho_i\}$, which is a graded manifold with an odd symplectic form of degree 2. Because E is a trivial vector bundle, $C^\infty(T^*[2,1]E[1]) = \Lambda^*\mathfrak{g}^* \otimes S^*\mathfrak{g} \otimes \Lambda^*\Gamma(TM)$ in a canonical way (otherwise we would need a connection to define a horizontal distribution on TE). The non-zero relations defining the Poisson bracket are, for $f \in C^\infty(M)$, $X \in \Gamma(TM)$, $\xi \in \mathfrak{g}^*$ and $v \in \mathfrak{g}$,

$$\{X, f\} = -\{f, X\} = X(f), \qquad \{v, \xi\} = -\{\xi, v\} = \xi(v).$$

Now $E[1]$ has a homological vector field (see Example 3.18) which lifts to $\mathcal{M}$ as explained in Example 3.32. That is, its lift is the homological vector field with Hamiltonian

$$S := \frac{1}{2}\sum_{i,j,k} c_{i,j}^k \xi^i \xi^j \rho_k + \sum_{i,a} \xi^i X_i^a p_a.$$

In an invariant form, $S = S_1 + S_2 \in (\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}) \oplus (\mathfrak{g}^* \otimes \Gamma(TM))$ acts on $\Lambda^2\mathfrak{g} \oplus \mathfrak{g}^*$ as $S_1(u, v, \xi) = \xi([u, v])$ and acts on $\mathfrak{g} \oplus T^*M$ as $S_2(v, \alpha) = \pm\alpha(X_v)$. The parities have been assigned in such a way that S is even and so, for any $H \in C^\infty(M)$, $S_H := S + H$ is again an even function giving rise to an odd Hamiltonian vector field. S_H solves the master equation and thus induces a homological vector field on $\mathcal{M}$ precisely when $\{S, H\} = 0$; that is, when H is invariant under the action of G.

If M was originally the space of fields of some field theory with an action functional H invariant by the action of a Lie group with Lie algebra $\mathfrak{g}$, then **BRST formalism** substitutes M by the data $(T^*[2,1]E[1], \omega, S_H)$, thought of as a model for the reduced space of fields. This is an example of a **BV manifold** [9]; that is, a supermanifold with an odd symplectic form and a Hamiltonia homological vector field. BV manifolds constitute the classical data of Batalin-Vilkovsky formalism. As explained in Section 1.2, AKSZ formalism is a field theory whose space of fields is the graded manifold $\underline{Mor}(T[1]\Sigma, \mathcal{M})$, for Σ an ordinary $(k+1)$-dimensional manifold and $(\mathcal{M}, \omega, Q)$ a symplectic Q-manifold with $w(\omega) = k$; in this formalism, $\underline{Mor}(T[1]\Sigma, \mathcal{M})$ is also a BV manifold with homological vector field constructed as in Example 3.20.

Example* 3.34 *($T^*[2] \oplus (E \oplus E^*)[1]$). Let $E \to M$ be an ordinary vector bundle with a connection $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ and consider the graded manifold

$$\mathcal{M}^\nabla = T^*[2] \oplus (E \oplus E^*)[1] = (M, \Gamma(\Lambda^*(E \oplus E^*) \otimes S^*TM)).$$

This is similar to Example 3.33, but in a non-trivial vector bundle and with different parities on the momenta. The connection ∇ (which we have not used yet) induces a metric connection still denoted by ∇ on $E \oplus E^*$ with its canonical pairing (which we denote in the following by $\langle\cdot,\cdot\rangle$) as

$$d(\xi(e)) = \xi(\nabla_X e) + (\nabla_X \xi)(e)$$

for $e \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$. This allows us to define the following relations on $C^\infty(\mathcal{M}^\nabla)$: For $f \in C^\infty(M)$, $s_1, s_2 \in \Gamma(E \oplus E^*)$ and $X, Y \in \Gamma(TM)$,

$$\begin{aligned} \{s_1, s_2\} &= \langle s_1, s_2\rangle, & \{X, f\} &= X(f), \\ \{X, s_1\} &= \nabla_X s_1, & \{X, Y\} &= [X, Y] - \Omega(X, Y), \end{aligned}$$

where $\Omega(X, Y) \in \Gamma(\Lambda^2(E \oplus E^*))$ is the curvature of the metric connection ∇. These relations extend in a unique way to a non-degenerate even Poisson bracket of degree -2 on $C^\infty(\mathcal{M}^\nabla)$, so this is also a symplectic graded manifold. We do not present here the proof because we will perform it in Proposition 4.13 below in a slightly more general context.

3.5. N-manifolds and Lie Algebroids

In this section we study a particular class of graded manifolds called **N-manifolds**. Their structure is such that phenomena as the one in Example 2.17 cannot happen and so they can indeed be thought as fiber bundles in some sense. However, in general they are not vector bundles and the best way to understand their structure is a combination of the graded approach and the fiber bundle one. We will also prove **Vaintrob's Theorem** characterizing Lie algebroids as NQ-manifolds of degree 1.

Definition 3.35. *An N-**manifold** $\mathcal{M}$ is a graded manifold in which there is only nonzero dimension on positive weights, and where parity equals weight mod 2. An NQ-**manifold** is an N-manifold with a homological vector field Q. A **symplectic NQ-manifold** is an NQ-manifold with a symplectic structure such that Q is Hamiltonian.*

In the context of N-manifolds we use the word ***degree*** for the weight of each element, which also identifies its parity, and we say that the degree of $\mathcal{M}$ is d if this is the maximum degree on which the dimension of $\mathcal{M}$ is nonzero. We denote by $\mathcal{A}^k = C^\infty_k(\mathcal{M})$ the space of degree k functions on $\mathcal{M}$ and by $\mathcal{A}_k$ the graded algebra that $\mathcal{A}^0$,...,$\mathcal{A}^k$ generate.

The main observation is that on an N-manifold one can only obtain degree k functions by operating with functions of degree less than or equal to k. Thus, for example, a degree 1 N-manifold $\mathcal{M} = (M, C^\infty(\mathcal{M}))$ is always isomorphic as a graded manifold to $E[1]$ for a canonical vector bundle $E \to M$. To see this we can go back to Definition 2.10 and see that we have a covering $\{U_\alpha\}_\alpha$ of M and morphisms $\varphi_\alpha : C^\infty(\mathcal{U}_\alpha) \to C^\infty(V_\alpha) \otimes \Lambda^*\mathbb{R}^m$ (where $\mathcal{U}_\alpha = (U_\alpha, C^\infty(\mathcal{M})_{|U_\alpha}))$ such that the coordinate changes induce isomorphisms of graded algebras

$$\varphi_{\alpha\beta} : C^\infty(V_\beta) \otimes \Lambda^*\mathbb{R}^m \to C^\infty(V_\alpha) \otimes \Lambda^*\mathbb{R}^m.$$

Now such an isomorphism must respect the degrees, and the only way to do this is by sending $C^\infty(V_\beta)$ functions to $C^\infty(V_\alpha)$ functions and applying a linear transformation (possibly depending on the degree 0-part) to the degree 1 coordinates, which defines precisely the transition morphisms of a vector bundle E with $\mathcal{M} \cong E[1]$, as claimed. If the degree of an N-manifold $\mathcal{M}$ is 2, then the coordinate transformations

$$\varphi_{\alpha\beta} : C^\infty(V_\beta) \otimes \Lambda^* \mathbb{R}^{m_1} \otimes S^* \mathbb{R}^{m_2} \to C^\infty(V_\alpha) \otimes \Lambda^* \mathbb{R}^{m_1} \otimes S^* \mathbb{R}^{m_2}.$$

must respect the preceeding remarks in degree 0 and 1, but have slightly more freedom in degree 2. Namely, one can obtain a degree 2 function by multiplying two degree 1 functions. This translates to the fact that $\mathcal{M}_1 := (M, \mathcal{A}_1)$ is an N-manifold of degree 1, and $\mathcal{M}$ can be seen as an affine bundle over $\mathcal{M}_1$, in a similar way as in Example 2.13 but with affine coordinate changes $A_{\alpha,\beta} : \mathbb{R}^{m_2} \to C^\infty(V_\alpha) \otimes \Lambda^* \mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}$ sending each coordinate $z^i \in \mathbb{R}^{m_2}$ to $\lambda^i_{r,s} \xi^r \xi^s + \mu^i_j z^j$ for some $\lambda_{r,s}, \mu_{i,j} \in C^\infty(V_\alpha)$, where $\{\xi^r\}_r$ are coordinates on $\mathbb{R}^{m_1}$. If $E \to M$ is the vector bundle such that $\mathcal{A}^1 = \Gamma(E^*)$, we can also see $\mathcal{A}^2$ as sections of an affine bundle over the ordinary manifold $\Lambda^2 E^*$. For a general N-manifold $\mathcal{M}$ of degree d, what we have is a tower of fibrations

$$\mathcal{M} = \mathcal{M}_d \to \mathcal{M}_{d-1} \to \dots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 = M,$$

where $\mathcal{M}_k := (M, \mathcal{A}_k)$ and each arrow represents an affine bundle projection; except for the last one, which is a vector bundle projection. Thus, $\mathcal{M} \to M$ is a polynomial bundle. Notice $\mathcal{A}^k = \mathcal{A}_k / \mathcal{A}_{k-1}$, and the above tower corresponds to the filtering

$$C^\infty(\mathcal{M}) = \mathcal{A}_d \supset \mathcal{A}_{d-1} \supset \dots \supset \mathcal{A}_2 \supset \mathcal{A}_1 \supset \mathcal{A}_0 = C^\infty(M).$$

We also notice that these $\mathcal{A}^k$ and $\mathcal{A}_k$ are locally free sheaves of $C^\infty(M)$-modules and thus they correspond to the sheaf of sections of some ***vector*** bundle over M in a non-canonical way.

In [59], Voronov constructs a **canonical linear model** for $\mathcal{M}$. This is a vector bundle $E \to M$ containing in some sense all the information of $\mathcal{M}$. Its sections can be canonically identified with $Der_{<0}\, C^\infty(\mathcal{M})$ (note $Der_{<0}\, S^* V^*$ is finite-dimensional as an $\mathbb{R}$-vector space for V non-negatively graded). This can be used to interpret a homological vector field on $\mathcal{M}$ via the algebraic operations that it defines on E through higher derived brackets, as in the construction of the brackets on an L_∞-algebra in Section 3.2. Recall that one of the key ideas in that construction was that $V[1] \cong Der_{-1}\, S^*(V[1])^*$ for non-graded V, so this is a natural generalization. The main motivation for this study is Vaintrob's Theorem 3.37, which we prove below. Before that we need Definition 3.36.

Definition 3.36. *A* **Lie algebroid** *is a vector bundle $E \to M$ over an ordinary manifold M endowed with a $C^\infty(M)$-linear* **anchor** *$a : \Gamma(E) \to \Gamma(TM)$ and a $\mathbb{R}$-linear* **bracket** *$[\cdot,\cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ such that:*

1. *$(\Gamma(E), [\cdot,\cdot])$ is pointwise a Lie algebra.*
2. *$[e_1, f e_2] = a(e_1)(f) e_2 + f[e_1, e_2]$ for $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(E)$.*

An important consequence of this definition is that $a([e_1, e_2]) = [a(e_1), a(e_2)]$ for $e_1, e_2 \in \Gamma(E)$, which follows easily from the Jacobi identity and Leibniz's rule (we present this computation in a similar context in Proposition 4.3 below).

Trivial examples of Lie algebroids are TM (or any integrable distribution on TM) with the identity as anchor and the ordinary bracket of vector fields or bundles of Lie algebras with null anchor. For $\mathfrak{g}$ a Lie algebra, $TM \oplus (M \times \mathfrak{g})$ is also a Lie algebroid with anchor the projection onto TM and bracket

$$[X \oplus v, Y \oplus u] := [X,Y] \oplus (X(u) - Y(v) + [v,w]), \qquad X,Y \in \Gamma(TM),\ u,v \in \Gamma(M \times \mathfrak{g}).$$

More interestingly, if G is a Lie group acting on a manifold M, then the trivial vector bundle $M \times \mathfrak{g}$ is a Lie algebroid with anchor the map sending each $v \in \mathfrak{g}$ to its fundamental vector field X_v. As we saw in Example 3.18, this structure can be interpreted in terms of a homological vector field. It turns out that this is always the case:

Theorem 3.37 (Vaintrob, [53]). *For any vector bundle $E \to M$, there is a one-to-one correspondence between the following objects.*

1. *Lie algebroid structures on E.*
2. *Homological vector fields of degree 1 on $E[1]$.*
3. *Poisson structures of degree -1 on $E^*[1]$.*

In particular,there is a one-to-one correspondence between Lie algebroids, NQ-manifolds of degree 1 and Poisson N-manifolds of degree 1 with Poisson bracket of degree -1.

Proof.
The last part will follow from the rest because we already know that an N-manifold of degree 1 corresponds to a vector bundle $E \to M$ in such a way that $C^\infty(\mathcal{M}) = \Gamma(\Lambda^* E^*)$; that is, $\mathcal{M} = E[1]$. In particular, derivations of $C^\infty(\mathcal{M})$ of degree -1 are determined by their restrictions $X : \Gamma(E^*) \to C^\infty(M)$, which must satisfy $X(fe) = fX(e)$ for $f \in C^\infty(M)$, $e \in \Gamma(E^*)$. In other words, as it was the case with L_∞-algebras, $Der_{-1}\, C^\infty(\mathcal{M}) = \Pi\,\Gamma(E)^{**} = \Pi\,\Gamma(E)$ is an abelian subalgebra of $Der\, C^\infty(\mathcal{M})$.

A homological vector field Q on $E[1]$ induces a derived bracket $[\cdot,\cdot]_Q$ on $Der\,\Gamma(\Lambda^* E^*)$ which leaves $\Pi\Gamma(E)$ invariant if $deg\, Q = 1$. Applying Proposition 3.2, we obtain an ordinary Lie algebra structure $[\cdot,\cdot]_Q$ on $\Gamma(E)$ via

$$[e_1, e_2]_Q = -[[Q, e_1], e_2].$$

The anchor is similarly defined: notice that for $e \in \Gamma(E)$ we have $-[Q,e] \in Der_0\, C^\infty(\mathcal{M})$ and thus $-[Q,e]$ restricts to an ordinary derivation of $C^\infty(M)$, which we call $a_Q(e)$. Leibniz's rule for Lie algebroids is now immediate from the properties of the Lie bracket:

$$[e_1, fe_2]_Q = -[[Q,e_1], fe_2] = -[Q,e_1](f)e_2 - f[[Q,e_1],e_2] = a_Q(e_1)(f)e_2 + f[e_1,e_2]_Q.$$

Conversely, given a Lie algebroid structure $([\cdot,\cdot]_E, a_E)$ on E, we define a $Q \in Der\,\Gamma(\Lambda^* E^*)$ of degree 1 by

$$Q(f)(e) = -a_E(e)(f),$$
$$Q(\xi)(e_1,e_2) = \xi([e_1,e_2]_E) - a_E(e_1)(\xi(e_2)) + a_E(e_2)(\xi(e_1)),$$

for $f \in C^\infty(M)$ and $\xi \in \Gamma(E^*)$. It is quickly verified that $Q(f\xi) = Q(f)\xi + fQ(\xi)$, so this is indeed a well-defined derivation. Moreover,

$$a_Q(e)(f) = -[Q, e](f) = -e(Q(f)) = -Q(f)(e) = a_E(e)(f).$$

And, for any $\xi \in \Gamma(E^*)$,

$$\begin{aligned} \xi([e_1, e_2]_Q) &= -\xi([[Q, e_1], e_2]) = -[[Q, e_1], e_2](\xi) = -(Qe_1 + e_1Q)e_2(\xi) + e_2(Qe_1 + e_1Q)(\xi) \\ &= -Q(\xi(e_2))(e_1) + Q(\xi(e_1))(e_2) + e_2(e_1(Q(\xi))) \\ &= a_E(e_1)(\xi(e_2)) - a_E(e_2)(\xi(e_1)) + Q(\xi)(e_1, e_2) = \xi([e_1, e_2]_E). \end{aligned}$$

This shows that the constructions we have given are inverse to each other. If $\{e_i\}_i$ are a local basis of sections of E with dual basis $\{\xi^i\}$, the way to write Q in coordinates is

$$Q = -\frac{1}{2} \sum_{i,j,k} c^k_{i,j} \xi^i \xi^j \frac{\partial}{\partial \xi^k} + \sum_i \xi^i a_E(e_i),$$

where $[e_i, e_j]_E = c^k_{i,j} e_k$. Notice the similarity with Examples 3.18 and 3.33. Thus, we see that $Q^2 = 0$ because, for $f \in C^\infty(M)$,

$$\begin{aligned} Q^2(f)(e_1, e_2) &= Q(f)([e_1, e_2]_E) - a_E(e_1)(Q(f)(e_2)) + a_E(e_2)(Q(f)(e_1)) \\ &= -a_E([e_1, e_2]_E)(f) + a_E(e_1)(a_E(e_2)(f)) - a_E(e_2)(a_E(e_1)(f)) = 0 \end{aligned}$$

and Q acts on each element of the dual basis $\{\xi^i\}_i$ as the pointwise Chevalley-Eilenberg differential, so it squares to zero on these. This implies $Q^2 = 0$ in general because Q satisfies Leibniz's rule and $\Gamma(\Lambda^* E^*)$ is locally generated as an algebra by $C^\infty(M) \otimes span\{\xi^i\}_i$.

In order to complete the proof, we simply notice that Poisson brackets of degree -1 on $E^*[1]$ are determined by a Lie algebra structure on $\Gamma(E)$ and an action of $\Gamma(E)$ on $C^\infty(M)$ satisfying

$$\{e_1, fe_2\} = \{e_1, f\}e_2 + f\{e_1, e_2\},$$

which is then extended to all of $\Gamma(\Lambda^* E)$ through Leibniz's rule. This is precisely the data of a Lie algebroid structure on E if we define $a(e_1)f = \{e_1, f\}$ and $\{e_1, e_2\} = [e_1, e_2]$. □

Theorem 3.37 suggests different ways in which Lie algebroids can be generalized:

1. An ***L_∞-algebroid*** or **Lie n-algebroid** [46] is a vector bundle $E \to M$ with a homological vector field of arbitrary degree on $E[1]$. This amounts to a fiberwise L_∞-algebra structure and a sequence of multilinear anchors $E \otimes ... \otimes E \to TM$ satisfying Leibniz rules on each argument with appropriate signs.

2. A ***non-linear Lie algebroid*** [59] is an NQ-manifold $\mathcal{M}$ of arbitrary degree. As previously stated, these can be interpreted through the canonically associated vector bundle $Der_{<0} C^\infty(\mathcal{M}) \to M$, where Q defines algebraic operations satisfying a complicated list of axioms through derived brackets.

3. A k-**fold Lie algebroid** (or double, triple, etc. Lie algebroid) [56] is a k-fold vector bundle $\mathcal{E}$ as in Example 2.15 with k different vector fields $Q_1, ..., Q_k$ such that $w_i(Q_j) = \delta_{ij}$ for w_i the weights on $\mathcal{E}$ and $[Q_i, Q_j] = 0$, for $i, j = 1, ..., k$. For $k = 2$ this is equivalent to a commuting square of vector bundles with Lie algebroid structures on the top and left arows that are compatible in a sense that will be discussed in detail in Examples 4.8 and 4.25.

These three notions are closely related. A k-fold Lie algebroid is a particular case of a non-linear Lie algebroid, since $\mathcal{E}$ is an NQ-manifold with grading $w_1 + ... + w_k$ and homological vector field $Q_1 + ... + Q_k$ (in fact, one may take any other linear combination). Moreover, as we know, an NQ-manifold $\mathcal{M} \to M$ can be identified in a non canonical way with a vector bundle $E \to M$, as it happens with every graded manifold by Batchelor's Theorem. It turns out that the homological vector field Q induces a non canonical L_∞-algebroid strucure on E, only defined up to L_∞-morphism [59]. In some sense which is still not clearly understood, non-linear Lie algebroids and L_∞-algebroids contain the same information.

In the next chapter we will present Courant algebroids and prove Roytenberg's Theorem characterizing them as symplectic NQ-manifolds of degree 2. Their relation with L_∞-algebras is described in Remark 4.2, and Courant algebroids arising as double Lie algebroids are studied in Examples 4.8 and 4.25. In any case, Courant algebroids can be seen as a higher analog of Lie algebroids, and this point of view seems to be useful in order to extend some of the work on Lie algebroids to Courant algebroids, such as representation theory [7], Chern-Simmons theory [49] or integration to Lie groupoids [51].

CHAPTER 4

The Ševera-Roytenberg correspondence

In this chapter we prove Roytenberg's Theorem on the correspondence between Courant algebroids and symplectic NQ-manifolds of degree 2 and we interpret some objects from generalized Riemannian geometry in terms of this correspondence. In Section 4.1 we introduce **Courant algebroids** and present the first examples. In Section 4.2 we start the proof of the Ševera-Roytenberg correspondence by characterizing **symplectic N-manifolds** of degree 1 and 2 as ordinary manifolds M and pseudo-Eculidean vector bundles E, respectively. The proof is concluded in Section 4.3, where it is shown that an additional **Hamiltonia homological vector field** on the corresponding graded manifolds is equivalent to a structure of, respectively, Poisson manifold on M and Courant algebroid on E. In Section 4.4 we show how the language of graded geometry uncovers some properties of Courant algebroids by studying the concrete examples of **exact** Courant algebroids and **Drinfeld doubles** of Lie bialgebroids. Section 4.5 serves as an introduction to the basic tools of **generalized Riemannian geometry** as introduced by Hitchin [26] and in Section 4.6 we construct a graded Poisson manifold which allows to interpret notions of generalized Riemannian geometry from a new perspective. In particular, we obtain **Bianchi identities** for the curvature of a generalized connection in this way.

4.1. Courant Algebroids

In this section we introduce the notion of **Courant algebroid** and we present the main examples of such objects. Courant algebroids are the central object in generalized geometry: if differential geometry studies differentiable manifolds M primarily through constructions on its tangent space TM, generalized geometry uses the bundle $TM \oplus T^*M$ instead, the motivation being second-dimensional variational problems, as explained in Section 1.1. This vector bundle has a rich structure which is encoded in the axioms of Courant algebroids.

Definition 4.1. *A* **Courant algebroid** *is a vector bundle $E \to M$ endowed with $C^\infty(M)$-linear operations $a : \Gamma(E) \to \Gamma(TM)$ (the* **anchor***) and $\langle\cdot,\cdot\rangle : \Gamma(E)\otimes\Gamma(E) \to C^\infty(M)$ (the* **pairing***) and an $\mathbb{R}$-linear operation $[\cdot,\cdot] : \Gamma(E)\otimes\Gamma(E) \to \Gamma(E)$ (the* **Dorfman bracket***) such that, for $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$,*

1. $\langle\cdot,\cdot\rangle$ *is symmetric and non-degenerate,*
2. $a(e_1)(\langle e_2, e_3\rangle) = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,
3. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
4. $[e_1, f e_2] = a(e_1)(f)e_2 + f[e_1, e_2]$,
5. $[e_1, e_2] + [e_2, e_1] = d_E\langle e_1, e_2\rangle$,

where $d_E := \kappa\circ a^* d$ *for* d *the exterior derivative,* a^* *the dual map of* a *and* $\kappa : E \to E^*$ *the isomorphism induced by* $\langle\cdot,\cdot\rangle$*. A* ***morphism of Courant algebroids*** $E_1 \to M_1$ *and* $E_2 \to M_2$ *is a pair* (f, φ) *with* $f : M_1 \to M_2$ *a* C^∞ *map and* $\varphi : E_1 \to f^*E_2$ *an orthogonal* $C^\infty(M)$*-linear map preserving* $[\cdot,\cdot]$ *and* a*.*

In the following we shall identify $\Gamma(E)$ and $\Gamma(E^*)$ through $\langle\cdot,\cdot\rangle$ without explicitely mentioning the isomorphism κ.

***Remark* 4.2.** In some early works such as [11], [37], [41], Courant algebroids were defined in terms of the skew-symmetrization of the Dorfman bracket $[\cdot,\cdot]$

$$[\![e_1, e_2]\!] := \frac{1}{2}\left([e_1, e_2] - [e_2, e_1]\right) = [e_1, e_2] - \frac{1}{2}d_E\langle e_1, e_2\rangle,$$

which is called the **Courant bracket**. It is now generally accepted that the Dorfman bracket $[\cdot,\cdot]$ is more fundamental: it satisfies the Jacobi identity and a Leibniz rule, the pairing on E is $[\cdot,\cdot]$-invariant in the sense of 2 and it appears naturally in the Ševera-Roytenberg correspondence. However, it is interesting to note that the defect of the Courant bracket $[\![\cdot,\cdot]\!]$ on satisfying the Jacobi identity is

$$[\![[\![e_1, e_2]\!], e_3]\!] - [\![[\![e_1, e_3]\!], e_2]\!] + [\![[\![e_2, e_3]\!], e_1]\!] = -\frac{1}{2}d_E\left(\langle[\![e_1, e_2]\!], e_3\rangle - \langle[\![e_1, e_3]\!], e_2\rangle + \langle[\![e_2, e_3]\!], e_1\rangle\right),$$

which is obtained by writing

$$[\![[\![e_1, e_2]\!], e_3]\!] = [[\![e_1, e_2]\!], e_3] - \frac{1}{2}d_E\langle[\![e_1, e_2]\!], e_3\rangle$$

and using the Jacobi identity for $[\cdot,\cdot]$. If we consider the graded vector space $V = C^\infty(M)[1]\oplus\Gamma(E)$ and define superskew-symmetric $\mathbb{R}$-linear maps $l_k : V^{\otimes^k} \to V$ for $k = 1, 2, 3$, $f \in C^\infty(M)$ and $e_1, e_2, e_3 \in \Gamma(E)$ as

$$\begin{aligned} l_1(f) &= d_E(f), \\ l_2(e_1, f) &= a(e_1)(f), \\ l_2(e_1, e_2) &= [\![e_1, e_2]\!], \\ l_3(e_1, e_2, e_3) &= -\frac{1}{6}\underset{1,2,3}{\mathrm{cycl}}\langle[\![e_1, e_2]\!], e_3\rangle \end{aligned}$$

and zero otherwise, then we claim that l_1, l_2, l_3 are the brackets of an L_∞-algebra structure on V. If J^r is the rth jacobiator of the brackets l_j, the computation above says precisely that $J^3(e_1, e_2, e_3) = 0$. The fact that $J^1 = 0$ is simply $d_E^2 = 0$, which is obvious, while $J^2 = 0$

and $J^3(f_1, e_2, e_3) = 0$ follow easily from Property 3 in Lemma 4.3 below. Then $J^4 = 0$ is obtained by writing the above measurement of the Jacobi identity for $[\![\cdot,\cdot]\!]$ as

$$l_3(l_2(e_1,e_2),e_3,e_4) - l_3(l_2(e_1,e_3),e_2,e_4) + l_3(l_2(e_2,e_3),e_1,e_4) = -l_2(l_3(e_1,e_2,e_3),e_4)$$

and $J^5 = 0$ is a bit more technical; the proof can be found in [41]. Recall that for finite-dimensional L_∞-algebras we constructed a corresponding Q-manifold over a point; the analog for this infinite-dimensional L_∞-algebra is the Q-manifold over M that is given by Roytenberg's Theorem 4.21.

***Lemma* 4.3.** Let $E \to M$ be a Courant algebroid. Then, the following properties are satisfied for $e_1, e_2 \in \Gamma(E)$, $f, g \in C^\infty(M)$ and $v_1, v_2 \in \Gamma(T^*M)$:

1. $[a(e_1), a(e_2)] = a([e_1, e_2])$.
2. $aa^*v_1 = 0$. In particular, $\langle a^*v_1, a^*v_2\rangle = [a^*v_1, a^*v_2] = 0$.
3. $[e_1, a^*v_1] = a^*\mathcal{L}_{a(e_1)}v_1$.
4. $[a^*v_1, e_1] = -a^*\iota_{a(e_1)}dv_1$
5. $[fe_1, ge_2] = fg[e_1,e_2] + fa(e_1)(g)e_2 - ga(e_2)(f)e_1 + \langle e_1, ge_2\rangle d_E f$,

where $\mathcal{L}_X$ is the Lie derivative along X and ι_Y is the contraction with Y.

Proof.
From 4 and 3 of Definition 4.1 we obtain

$$\begin{aligned}
a([e_1,e_2])(f)e_3 &= [[e_1,e_2], fe_3] - f[[e_1,e_2],e_3] \\
&= [e_1,[e_2,fe_3]] - [e_2,[e_1,fe_3]] - f[e_1,[e_2,e_3]] + f[e_2,[e_1,e_3]] \\
&= [e_1, a(e_2)(f)e_3 + f[e_2,e_3]] - [e_2, a(e_1)(f)e_3 + f[e_1,e_3]] - f[e_1,[e_2,e_3]] + f[e_2,[e_1,e_3]] \\
&= [a(e_1),a(e_2)](f)e_3 + a(e_2)(f)[e_1,e_3] + [e_1, f[e_2,e_3]] \\
&\quad - a(e_1)(f)[e_2,e_3] - [e_2, f[e_1,e_3]] - f[e_1,[e_2,e_3]] + f[e_2,[e_1,e_3]] \\
&= [a(e_1),a(e_2)](f)e_3.
\end{aligned}$$

Now in a local frame $\{\xi^i\}_i$ with dual frame $\{\tilde{\xi}^i\}_i$ (that is, $\langle \xi^i, \tilde{\xi}^j\rangle = \delta_{ij}$) we use 5 to obtain, for any $f \in C^\infty(M)$,

$$aa^*df = a(d_E\langle f\xi_1, \tilde{\xi}_1\rangle) = a([f\xi_1, \tilde{\xi}_1]) + a([\tilde{\xi}_1, f\xi_1]) = 0,$$

which implies $aa^* = 0$ in general by $C^\infty(M)$-linearity of the anchor. Then isotropy of a^* is immediate because $\langle a^*v_1, a^*v_2\rangle = v_1(a(a^*v_2))$. Notice then that

$$a([e_2, a^*v_1]) = [a(e_2), aa^*v_1] = 0, \qquad a([a^*v_1, e_2]) = [aa^*v_1, a(e_2)] = 0$$

and so, using 2 from Definition 4.1,

$$\langle [a^*v_1, a^*v_2], e_2\rangle = a(a^*v_1)(\langle a^*v_2, e_2\rangle) - \langle a^*v_2, [a^*v_1, e_2]\rangle = -v_2(a([a^*v_1,e_2])) = 0,$$

which implies $[a^*v_1, a^*v_2] = 0$. Using 2 again,

$$\begin{aligned}\langle [e_1, a^*v_1], e_2\rangle &= a(e_1)(\langle a^*v_1, e_2\rangle) - \langle a^*v_1, [e_1, e_2]\rangle = a(e_1)(v_1(a(e_2))) - v_1(a([e_1, e_2]))\\ &= \mathcal{L}_{a(e_1)}v_1(a(e_2))\end{aligned}$$

And then using 5,

$$\begin{aligned}\langle [a^*v_1, e_1], e_2\rangle &= -\langle [e_1, a^*v_1], e_2\rangle + a(e_2)(\langle a^*v_1, e_1\rangle) = -\mathcal{L}_{a(e_1)}v_1(a(e_2)) + a(e_2)(v_1(a(e_1)))\\ &= -dv_1(a(e_1), a(e_2)).\end{aligned}$$

The last property follows directly from 4 and 5 in Definition 4.1. □

Example 4.4 (Quadratic Lie Algebras). If $M = \{*\}$ is a point, then a Courant algebroid over M is a vector space V with a pseudo-Euclidean metric $\langle\cdot,\cdot\rangle$ and a Lie bracket $[\cdot,\cdot]$ such that

$$0 = \langle [u_1, u_2], u_3\rangle + \langle u_2, [u_1, u_3]\rangle.$$

This is a **quadratic Lie algebra**, as in Example 3.8. The pairing $\langle\cdot,\cdot\rangle$ can be seen as a symplectic form of degree 2 on $V[1]$; its corresponding Poisson bracket is the extension of $\langle\cdot,\cdot\rangle$ to Λ^*V^* through Leibniz's rule. Recall the **Cartan 3-form** $\mu \in \Lambda^3V^*$ defined by $\mu(u_1, u_2, u_3) = \langle [u_1, u_2], u_3\rangle$. It is interesting to notice that

$$\{\mu, u_3\}(u_1, u_2) = \mu(u_1, u_2, u_3) = \langle [u_1, u_2], u_3\rangle = d_{CE}u_3(u_1, u_2),$$

so the Chevalley-Eilenberg differential is Hamiltonian with respect to this symplectic structure and μ is its Hamiltonian function. In Example 3.8 we saw that $d_{CE}\mu = 0$, which now takes the form $\{\mu, \mu\} = 0$. Finally, notice that $[u_1, u_2] = \mu(u_1, u_2, \cdot)$, which means that the Lie bracket can be expressed in terms of μ as

$$[u_1, u_2] = \{\{\mu, u_1\}, u_2\}.$$

Roytenberg's Theorem generalizes these ideas to arbitrary Courant algebroids. Note that the L_∞-algebra constructed in Remark 4.2 coincides in this case with the string Lie 2-algebra from Example 3.8.

Example 4.5 (Dorfman Bracket). For M any C^∞ manifold, consider the vector bundle $E := TM \oplus T^*M \to M$. There is a canonical non-degenerate pairing on E; namely, for $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$,

$$\langle X + \alpha, Y + \beta\rangle := \alpha(Y) + \beta(X).$$

Moreover, there is a canonical choice for an anchor:

$$\begin{aligned}a : \Gamma(TM) \oplus \Gamma(T^*M) &\to \Gamma(TM)\\ X + \alpha &\mapsto X.\end{aligned}$$

The **Dorfman bracket** is the following operation on $\Gamma(E)$:

$$[X + \alpha, Y + \beta]_E = [X, Y] + \mathcal{L}_X\beta - \iota_Y d\alpha,$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields. This bracket is suggested by relations 3 and 4 from Lemma 4.3 and motivated by the discussion in Section 1.1. We claim that these operations endow E with the structure of a Courant algebroid. First, recall that $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ and compute

$$\begin{aligned}\langle [X+\alpha, Y+\beta]_E, Z+\gamma\rangle + \langle Y+\beta, [X+\alpha, Z+\gamma]_E\rangle = \\ &= \iota_{[X,Y]}\gamma + \iota_Z(\mathcal{L}_X\beta - \iota_Y d\alpha) + \iota_{[X,Z]}\beta + \iota_Y(\mathcal{L}_X\gamma - \iota_Z d\alpha) \\ &= \mathcal{L}_X\iota_Y\gamma + \mathcal{L}_X\iota_Z\beta = \iota_X d(\gamma(Y)+\beta(Z)).\end{aligned}$$

For the Jacobi identity we also recall $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ and we see

$$\begin{aligned}[[X+\alpha, Y+\beta]_E, Z+\gamma]_E + [Y+\beta, [X+\alpha, Z+\gamma]_E]_E = \\ &= [[X,Y] + \mathcal{L}_X\beta - \iota_Y d\alpha, Z+\gamma]_E + [Y+\beta, [X,Z] + \mathcal{L}_X\gamma - \iota_Z d\alpha]_E \\ &= [[X,Y],Z] + \mathcal{L}_{[X,Y]}\gamma - \iota_Z d(\mathcal{L}_X\beta - \iota_Y d\alpha) \\ &\quad + [Y,[X,Z]] + \mathcal{L}_Y(\mathcal{L}_X\gamma - \iota_Z d\alpha) - \iota_{[X,Z]} d\beta \\ &= [X,[Y,Z]] + \mathcal{L}_X\mathcal{L}_Y\gamma - \mathcal{L}_X\iota_Z d\beta - \mathcal{L}_Y\iota_Z d\alpha + \iota_Z d\iota_Y d\alpha \\ &= [X,[Y,Z]] + \mathcal{L}_X(\mathcal{L}_Y\gamma - \iota_Z d\beta) - \iota_{[Y,Z]} d\alpha \\ &= [X+\alpha, [Y+\beta, Z+\gamma]_E]_E.\end{aligned}$$

Then

$$\begin{aligned}[X+\alpha, f(Y+\beta)]_E &= [X, fY] + \mathcal{L}_X(f\beta) - \iota_{fY} d\alpha = f[X+\alpha, Y+\beta]_E + X(f)Y + X(f)\beta \\ &= a(X+\alpha)(Y+\beta) + f[X+\alpha, Y+\beta]_E.\end{aligned}$$

and finally

$$[X+\alpha, Y+\beta]_E + [Y+\beta, X+\alpha]_E = \mathcal{L}_X\beta - \iota_Y d\alpha + \mathcal{L}_Y\alpha - \iota_X d\beta = d(\beta(X)+\alpha(Y)),$$

which concludes the proof because in this case it is clear that $d_E = d$.

This example (or, more precisely, its skew-symmetrization) is the structure that Courant studied in [11] for a unified treatment of closed 2-forms (**presymplectic** forms) and Poisson tensors on M which eventually gave Courant algebroids their name, although previous work by Dorfman [14] had already dealt with these same ideas. **Almost Dirac structures** were defined as isotropic subbundles $L \subset TM \oplus T^*M$ with $rank(L) = dim\, M$ and **Dirac structures** as almost Dirac structures that are closed under the Dorfman bracket. Bivectors π and two-forms ω have naturally associated almost Dirac structures $L_\pi = \{\alpha + \iota_\alpha\pi \;:\; \alpha \in T^*M\}$ and $L_\omega = \{X + \iota_X\omega \;:\; X \in TM\}$, and it is not hard to see that L_π and L_ω are in fact Dirac structures if and only if $[\pi,\pi] = 0$ or $d\omega = 0$, respectively.

Definition 4.6. *A Courant algebroid E is* ***transitive*** *if its anchor is surjective. It is* ***exact*** *if the canonical sequence*

$$0 \to T^*M \xrightarrow{a^*} E \xrightarrow{a} TM \to 0$$

is exact. An ***almost Dirac structure*** *on E is a maximal isotropic subbundle $L \subset E$ and a* ***Dirac structure*** *on E is an almost Dirac structure that is closed under the Dorfman bracket. Equivalently, a Dirac structure is a subbundle $L \subset E$ such that $(L, a, [\cdot,\cdot]_E)$ is a Lie algebroid.*

Example* 4.7 *(Exact Courant Algebroids). As it is clear from the definition, any Courant algebroid structure over $TM \oplus T^*M$ with its canonical pairing and anchor is exact. In fact, the following argument shows that any exact Courant algebroid E over M is isomorphic to one of this form:

Choose a splitting $s_0 : TM \to E$ of the above exact sequence, consider $\rho \in S^2(\Gamma(T^*M))$ given by $\rho(X,Y) = \langle s_0(X), s_0(Y)\rangle$ and define $s : TM \to E$ by $s(X) = s_0(X) - \frac{1}{2}a^*\rho(X,\cdot)$. Then s is isotropic because so is a^*:

$$\begin{aligned}\langle s(X), s(Y)\rangle &= \langle s_0(X), s_0(Y)\rangle - \frac{1}{2}\rho(X, as_0(Y)) - \frac{1}{2}\rho(Y, as_0(X)) + \frac{1}{4}\langle a^*\rho(X,\cdot), a^*\rho(Y,\cdot)\rangle \\ &= \langle s_0(X), s_0(Y)\rangle - \langle s_0(X), s_0(Y)\rangle = 0.\end{aligned}$$

This means that we can define an isomorphism of vector bundles

$$\begin{aligned}\varphi : TM \oplus T^*M &\to E \\ X + \alpha &\mapsto s(X) + a^*\alpha\end{aligned}$$

which preserves the metric an the anchor of the canonical Courant algebroid structure on $TM \oplus T^*M$. In particular, φ relates the Courant algebroid structure $[\cdot,\cdot]_E$ on E with some Courant algebroid structure $[\cdot,\cdot]_\varphi$ on $TM \oplus T^*M$. Let us compute it explicitly:

$$\begin{aligned}\varphi([X+\alpha, Y+\beta]_\varphi) &= [s(X), s(Y)]_E + [s(X), a^*\beta]_E + [a^*\alpha, s(Y)]_E + [a^*\alpha, a^*\beta]_E \\ &= [s(X), s(Y)]_E + \varphi(\mathcal{L}_X\beta - \iota_Y d\alpha),\end{aligned}$$

where we have used 3 and 4 from Lemma 4.3. Finally, the relations

$$\begin{aligned}a(\varphi^{-1}([s(X), s(Y)]_E)) &= \varphi^{-1}(a([s(X), s(Y)]_E)) = \varphi^{-1}([as(X), as(Y)]_E) = [X,Y], \\ \langle \varphi^{-1}([s(X), s(Y)]_E), Z\rangle &= \langle [s(X), s(Y)]_E, s(Z)\rangle\end{aligned}$$

tell us that the induced bracket on $TM \oplus T^*M$ is

$$[X+\alpha, Y+\beta]_\varphi = [X,Y] + \mathcal{L}_X\beta - \iota_Y d\alpha + \iota_Y\iota_X H,$$

for $H(X,Y,Z) := \langle [s(X), s(Y)]_E, s(Z)\rangle$ (see the similarity with the Cartan 3-form in Example 4.4). It is not hard to prove that H is a totally skew-symmetric tensor; i.e., $H \in \Omega^3(M)$. In fact, $dH = 0$ and it can also be shown that any closed 3-form H defines a structure of Courant algebroid via the above formula. Although this can be shown directly, we will prove it in Example 4.24 with the language of graded manifolds, which will give us a nice interpretation of this result which generalizes to non-exact Courant algebroids.

Moreover, a famous result by Ševera [50] asserts that, if we define two Courant algebroids E_1, E_2 to be in the same **small isomorphism class** whenever there exists an isomorphism $(f,\varphi) : E_1 \to E_2$ such that $f \in Diff_0(M)$ (identity component of $Diff(M)$), then the small isomorphism classes of exact Courant algebroids are classified by the cohomology classes of the corresponding three-forms $[H] \in H^3(M,\mathbb{R})$. The ordinary isomorphism classes of exact Courant algebroids are parameterized by $H^3(M,\mathbb{R})/\Gamma$, where $\Gamma = Diff(M)/Diff_0(M)$ is the mapping class group of M [20].

Example 4.8 (The Double of a Lie Bialgebroid). Consider a Lie algebroid $A \to M$, as in Definition 3.36. From Theorem 3.37 we see that this gives a derivation d_A of $\Gamma(\Lambda^* A^*)$ and a Poisson structure $[\cdot,\cdot]_A$ of degree -1 on $\Gamma(\Lambda^* A)$ (for $A = TM$, d_A is simply the exterior derivative). The derivation d_A also allows to define Lie derivatives $\mathcal{L}^A_v$ for $v \in \Gamma(A)$ on $\Gamma(\Lambda^* A^*)$ as $\mathcal{L}^A_v \xi = (\iota_v d_A + d_A \iota_v)\xi$, $\xi \in \Gamma(A^*)$. A direct computation analogous to the ordinary one shows that this Lie derivative satisfies all desired properties such as $\iota_{[v_1,v_2]_A} = [\mathcal{L}^A_{v_1}, \iota_{v_2}]$ and $\mathcal{L}^A_{[v_1,v_2]} = [\mathcal{L}^A_{v_1}, \iota_{v_2}]$, which where the key in Example 4.5 to prove that the Dorfman bracket is a Courant algebroid structure on $TM \oplus T^*M$. Hence, we see that there is a Courant algebroid structure on $E = A \oplus A^*$ defined by

$$
\begin{aligned}
a_E(v_1 + \xi_1) &= a_A(v_1),\\
\langle v_1 + \xi_1, v_2 + \xi_2 \rangle &= \xi_1(v_2) + \xi_2(v_1),\\
[v_1 + \xi_1, v_2 + \xi_2]_E &= [v_1, v_2]_A + \mathcal{L}^A_{v_1}\xi_2 - \iota_{v_2} d_A \xi_1,
\end{aligned}
$$

for $v_i + \xi_i \in \Gamma(A \oplus A^*)$, $i = 1, 2$. Assume now that we have an additional Lie algebroid structure on A^*; then we have a derivation d_{A^*} of $\Gamma(\Lambda^* A)$ and a Poisson structure $[\cdot,\cdot]_{A^*}$ of degree -1 on $\Gamma(\Lambda^* A^*)$. A **Lie bialgebroid** is a pair (A, A^*) of Lie algebroids in duality as vector spaces such that d_A is a derivation of $[\cdot,\cdot]_{A^*}$; that is,

$$
d_A[\gamma, \eta] = [d_A \gamma, \eta] + (-1)^{p+1}[\gamma, d_A \eta], \qquad \gamma \in \Gamma(\Lambda^p A^*),\ \eta \in \Lambda^q \Gamma(A^*).
$$

For example, (A, A^*) is always a Lie bialgebroid if A^* is considered as a Lie algebroid with zero anchor and bracket. The **(Drinfeld) double of a Lie bialgebroid** is a Courant algebroid structure on $E := A \oplus A^* \to M$ defined by:

$$
\begin{aligned}
a_E(v + \xi) &= a_A(v) + a_{A^*}(\xi),\\
\langle v_1 + \xi_1, v_2 + \xi_2 \rangle &= \xi_1(v_2) + \xi_2(v_1),\\
[v_1 + \xi_1, v_2 + \xi_2]_E &= [v_1, v_2]_A + \mathcal{L}^{A^*}_{\xi_1} v_2 - \iota_{\xi_2} d_{A^*} v_1 + [\xi_1, \xi_2]_{A^*} + \mathcal{L}^A_{v_1}\xi_2 - \iota_{v_2} d_A \xi_1,
\end{aligned}
$$

The fact that this indeed defines a Courant algebroid structure will be proved in a simple way using the language of graded geometry in Example 4.25; this perspective will also allow for a simple proof that (A, A^*) is a Lie bialgebroid if and only if so is (A^*, A), it will show how we can twist this structure in a similar way as in Example 4.7 and it will give a simple characterization of Dirac structures on $A \oplus A^*$.

After Dorfman and Courant's study of Dirac structures Courant algebroids began to appear in different contexts. The study of Lie bialgebroids (A, A^*) led naturally to the above Courant algebroid structure on $A \oplus A^*$ in [39] and a explicit general definition was first given in [37]. As explained in Section 1.1, Courant algebroids arise in the study of two-dimensional variational problems, which was first noted by Ševera as explained in his letters to Alan Weinstein [50]. In these same letters he sketches many ideas that have later been developed in greater detail, such as Roytenberg's Theorem 4.21 or the relation between Courant algebroids and Poisson-Lie T-duality, which is a very active field of research in these days [19], [52], [49].

4.2. Symplectic N-manifolds of Degree 1 and 2

In this section we start the proof of the Ševera-Roytenberg correspondence between symplectic NQ-manifolds of degree 2 and Courant algebroids. Namely, we will study **symplectic N-manifolds** of degree 1 and 2 in detail and we will prove that they are equivalent, respectively, to ordinary manifolds and pseudo-Euclidean vector bundles.

Consider an N-manifold $\mathcal{M}$ with a symplectic form ω of degree k. We recall the remarks succeeding Definition 3.21, which state that the dimension of $\mathcal{M}$ on degree l equals its dimension on degree $-l+k$. In particular, since there are no coordinates of negative degree, the degree of $\mathcal{M}$ cannot exceed d. We also remind that ω induces a non-degenerate Poisson bracket on $C^\infty(\mathcal{M})$ of degree $-k$. If $k = 0$, we immediately see that $\mathcal{M}$ is a symplectic manifold in the ordinary sense.

Lemma* 4.9 *(Darboux Coordinates). Let $(\mathcal{M}, \omega)$ be a symplectic N-manifold of degree $k \in \{1, 2\}$. Then,

- If $k = 1$, then every $p \in M$ admits an open neighborhood $\mathcal{U}_p \subset \mathcal{M}$ and coordinates $\{x^i, p_i\}_i$ with $deg(x^i) = 0$, $deg(p_i) = 1$ such that $\omega_{|\mathcal{U}_p} = dp_i dx^i$.
- If $k = 2$, then every $p \in M$ admits an open neighborhood $\mathcal{U}_p \subset \mathcal{M}$ and coordinates $\{x^a, \xi^i, p_a\}_{i,a}$ with $deg(x^a) = 0$, $deg(\xi^i) = 1$, $deg(p_a) = 2$ such that $\omega_{|\mathcal{U}_p} = dp_a dx^a + g_{ij} d\xi^i d\xi^j$ for an invertible constant matrix $(g_{ij})_{ij}$.

Proof.
When $k = 1$, take local coordinates $\{y^i, p_i\}$ with $deg(y^i) = 0$ and $deg(p_i) = 1$. Any 2-form ω of degree 1 can be locally written as $\omega = f_{i,j}(y) dp_i dy^j$. Closedness of ω implies that for each fixed i the one-form $f_{i,j}(y)dy^j$ is closed and so, using Poincaré's Lemma, in a sufficiently small neighborhood $f_{i,j}(y)dy^j = dx^i$ for a function x^i, giving the desired result.

When $k = 2$, any **closed** 2-form of degree 2 can be written in local coordinates $\{y^a, \xi^i, p_a\}_i$ with $deg(y^a) = 0$, $deg(\xi^i) = 1$ and $deg(p_a) = 2$ as $\omega = f_{a,b}(y) d\eta^a dy^b + g_{i,j} d\xi^i d\xi^j + h_{a,i}(y)\xi^i dy^a d\xi^i$ but the non-degeneracy condition implies that $h_{a,i} = 0$ because the odd variables ξ^i are not invertible. It also implies that $(g_{i,j})_{i,j}$ is invertible and, as before, in a sufficiently small neighborhood we have $f_{a,b}(y)dy^b = dx^a$, which concludes the proof. $\square$

***Remark* 4.10.** Darboux's Theorem for symplectic graded manifolds is true in greater generality — see [6] — but a complete proof is too tedious and not necessary for our interests.

We proceed to study symplectic N-manifolds $(\mathcal{M}, \omega)$ with $deg(\omega) = 1$. An example of such manifolds has been presented in Example 3.31, and we claim that every other example is isomorphic to this one. Indeed, we know that $\mathcal{M} = E^*[1] = (M, \Gamma(\Lambda^* E))$ for some vector bundle $E \to M$, and the existence of ω implies $rank(E) = dim(M)$. This symplectic structure gives a degree -1 Poisson bracket so, according to Vaintrob's Theorem 3.37, E has a Lie algebroid structure with anchor

$$\begin{aligned} a : \Gamma(E) &\to \Gamma(TM) \\ e &\mapsto \{e, \cdot\}. \end{aligned}$$

Since $\mathcal{M}$ is symplectic, the Poisson bracket is non-degenerate. This implies that the anchor is an isomorphism of vector bundles. Indeed, $\{e, f\} = 0\, \forall f \in C^\infty(M)$ implies $\{H, f\} = 0$ $\forall H \in C^\infty(\mathcal{M})$ because this algebra is locally generated by functions of degree 0 and 1 (and $\{\mathcal{A}^0, \mathcal{A}^0\} = 0$), and this means that the anchor is surjective. Since both vector bundles have the same dimension, a is an isomorphism of vector bundles. The Jacobi identity for the Poisson bracket implies that a is in fact an isomorphism of Lie algebras. Using Leibniz's rule it can be extended in a unique way to an isomorphism of Gerstenhaber algebras $\Gamma(\Lambda^* E) \to \Gamma(\Lambda^* TM)$ and so we obtain a canonical isomorphism of graded manifolds preserving the symplectic structure between $\mathcal{M}$ and $T^*[1]M$ (as in Example 3.31). In terms of local Darboux coordinates $\{x^i, p_i\}_i$ on $\mathcal{M}$ such that

$$\omega = dp_i dx^i,$$

the above isomorphism sends $x_i \mapsto x_i$ and $p_i \mapsto \partial_{x_i}$, as can be seen from Remark 3.28. It is also clear that any symplectomorphism $T^*[1]M \to T^*[1]N$ is uniquely determined by its restriction to $M \to N$. So we have proved:

***Proposition* 4.11.** If $(\mathcal{M}, \omega)$ is a symplectic N-manifold with $deg(\omega) = 1$, then $\mathcal{M}$ is canonically symplectomorphic to $T^*[1]M$, for M the underlying topological space of $\mathcal{M}$. In particular, the category of symplectic N-manifolds of degree 1 with symplectomorphisms is equivalent to the category of ordinary manifolds with diffeomorphisms.

***Remark* 4.12.** Now that we have a better understanding of symplectic N-manifolds we can study the isotropic (resp. coisotropic, etc.) submanifolds of $(T^*[1]M, \omega)$, as anticipated in Example 3.31. Any submanifold $i : \mathcal{N} \to T^*[1]M$ must be an N-manifold of degree 0 or 1; hence, an ordinary manifold N or $E[1]$ for a vector bundle $E \to N$ over an ordinary manifold N. Moreover, since $i : \mathcal{N} \to T^*[1]M$ must preserve the degrees, $j : N \to M$ is a submanifold of M and E is a subbundle of T^*M.

The property of being isotropic (resp. coisotropic, etc.) is local, so for $p \in N$ we may take Darboux coordinates on $T^*[1]M$ in $j(p) \in U \subset M$ and we can think of $T^*[1]M_{|U}$ as $\mathbb{R}^{2n}$ with its canonical symplectic structure; hence identifying $T_{j(p)}T^*[1]M$ with $T_{j(p)}[1]M \oplus T^*_{j(p)}[1]M$ (the notation is meant to indicate that these are $\Lambda^*\mathbb{R}^n$-modules), $T_pE[1]$ with $T_p[1]N \oplus E_p[1]$ and $\omega_{|j(p)}$ with the canonical pairing. Thus the symplectic complement of $T_pE[1]$ consists on (the pull-back of) vectors tangent to M which annihilate E and covectors on M vanishing on TN. To sum up, this shows that for every submanifold $\mathcal{N} \to T^*[1]M$ there exists a submanifold $N \to M$ such that (write $F_N := j^*T^*M/T^*N$ for the **conormal bundle** of N):

1. If $\mathcal{N}$ is isotropic, then $\mathcal{N} = E[1]$ for $E \to N$ a vector bundle with $E \subset F_N$ (including $E = \{0\}$).
2. If $\mathcal{N}$ is coisotropic, then $\mathcal{N} = E[1]$ for $E \to N$ a vector bundle with $F_N \subset E$.
3. If $\mathcal{N}$ is Lagrangian, then $\mathcal{N} = F_N[1]$.
4. If $\mathcal{N}$ is symplectic, then $\mathcal{N} = T^*[1]N$.

Let us now consider symplectic N-manifolds $(\mathcal{M}, \omega)$ with $deg(\omega) = 2$. We have an example of such graded manifolds in Example 3.34, and although this is not the only possible example, it will be useful to keep it in mind during the following argument. The Poisson bracket on $\mathcal{M}$ satisfies the relations

$$\{\mathcal{A}^0, \mathcal{A}^0\} = \{\mathcal{A}^1, \mathcal{A}^0\} = 0, \quad \{\mathcal{A}^1, \mathcal{A}^1\} \subset \mathcal{A}^0, \quad \{\mathcal{A}^2, \mathcal{A}^j\} \subset \mathcal{A}^j \text{ for } j = 0,\, 1,\, 2.$$

Since $\mathcal{A}^1 = \Gamma(E^*)$ and $\mathcal{A}^0 = C^\infty(M)$ for some vector bundle $E \to M$, we can think of the Poisson bracket in $\mathcal{A}^1$ as a fiberwise symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E^* which is moreover non-degenerate (in local Darboux coordinates $\{x^a, \xi^i, p_a\}$ with $\omega = dp_a dx^a + g_{i,j} d\xi^i d\xi^j$, the matrix g_{ij} defines this pairing) and so it lets us identify E and E^*. It extends to the whole of $\mathcal{A}_1$ through Leibniz's rule and so $\mathcal{M}_1 = E[1] = (M, \mathcal{A}_1)$ is a Poisson N-manifold, with $\mathcal{M}_2 \to \mathcal{M}_1$ a Poisson map. We claim that the whole structure of $\mathcal{M}$ is given by the data $(E, \langle \cdot, \cdot \rangle)$. To see this, we consider the anchor $a : \mathcal{A}^2 \to \Gamma(TM)$, $D \mapsto \{D, \cdot\}$, which is surjective by non-degeneracy as before and has $\mathcal{A}^1\mathcal{A}^1$ as its kernel. This is easily seen by noting that, in local Darboux coordinates, this map is

$$\sum_{i,j} f_{i,j}(x)\xi^i \xi^j + \sum_a h^a(x) p_a \mapsto \sum_a h^a(x) \frac{\partial}{\partial x^a},$$

as can be seen from Remark 3.28. On the other hand, elements $D \in \mathcal{A}^2$ also act on $\mathcal{A}^1 = \Gamma(E)$ through the Poisson bracket as first-oder differential operators satisfying

$$a(D)\langle e_1, e_2 \rangle = \langle De_1, e_2 \rangle + \langle e_1, De_2 \rangle$$

by the Jacobi identity. It follows from the non-degeneracy of the Poisson bracket that $D \in \mathcal{A}^2$ is determined by its action on $\mathcal{A}^1 = \Gamma(E)$ and $\mathcal{A}^0 = C^\infty(M)$, so it can be identified with a covariant first-order differential operator on E. Conversely, we claim that all covariant differential operators on E preserving the inner product are represented by functions of $\mathcal{A}^2$. To see this, note that the preceeding remarks imply that there is an exact sequence

$$0 \to \Gamma(\Lambda^2 E^*) \to \mathcal{A}^2 \to \Gamma(TM) \to 0,$$

Now $\Gamma(\Lambda^2 E^*)$ is the space of skew-symmetric endomorphisms of E, which can be thought of as sections of the vector bundle $OF(E) \times_{O(k)} \mathfrak{so}(k)$, where $OF(E)$ is the principal $O(k)$-bundle of orthogonal frames of E, $k = rank(E)$ and we are considering the adjoint action of $O(k)$ in $\mathfrak{so}(k)$ for the fibre product. In particular, by identifying $\mathcal{A}^2$ with differential operators on E we see that the above sequence is the ***Atiyah exact sequence*** [1] of the principal bundle $OF(E)$. Thus $\mathcal{A}^2$ coincides with the space of sections of the Atiyah Lie algebroid $\mathbb{A} = TOF(E)/G$, which represents precisely covariant differential operators on E preserving $\langle \cdot, \cdot \rangle$. We conclude that $C^\infty(\mathcal{M}) = \Gamma(\Lambda^* E^* \otimes S^* \mathbb{A})/I$, where I is the homogeneous ideal generated by $\tau \otimes 1 - 1 \otimes \tau$, for $\tau \in \Gamma(\Lambda^2 E^*)$.

It also follows that, for $\mathcal{N}$ another symplectic N-manifold of degree 2, a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ preserving the symplectic structure is determined by its restriction $\varphi_{|\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{N}_1$, because the action of the CDO's on degree 2 must be preserved. Moreover,

[1]The Atiyah exact sequence of a principal bundle $\pi : P \to M$ with structure group G is the following exact sequence of Lie algebroids:

$$0 \to P \times_G \mathfrak{g} \to TP/G \to TM \to 0,$$

which can be obtained as the quotient by G of the exact sequence $0 \to Ker\, d\pi \to TP \to \pi^* TM \to 0$.

if $\mathcal{M}_1 = E_{\mathcal{M}}[1]$ and $\mathcal{N}_1 = E_{\mathcal{N}}[1]$, then $\varphi_{|\mathcal{M}_1}$ is determined by a metric-preserving map $\Gamma(E_{\mathcal{N}}) \to \Gamma(E_{\mathcal{M}})$. So there is a contravariant functor sending each pseudo-Euclidean vector bundle to its corresponding graded manifold.

We now want to construct, for a given pseudo-Euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$, a symplectic N-manifold $\mathcal{M}$ of degree 2 such that $\mathcal{A}^1 = \Gamma(E^*)$ and that the Poisson bracket on $\mathcal{A}^1$ coincides with the pairing $\langle\cdot,\cdot\rangle$. From the preceeding discussion, it will automatically follow that $\mathcal{A}^2$ can be identified with sections of the Atiyah algebroid of E. Recall Example 3.34. The same idea can be applied here: Take a **metric** connection on E with curvature form $\Omega \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 E^*)$ given by $\Omega(X,Y)(e_1,e_2) = \langle \nabla_X\nabla_Y e_1 - \nabla_Y\nabla_X - \nabla_{[X,Y]}e_1, e_2\rangle$, consider $\mathcal{M}^\nabla = T^*[2]M \oplus E[1] = (M, \Gamma(\Lambda^* E \otimes S^*TM))$ and define

$$\{e_1,e_2\} = \langle e_1,e_2\rangle, \qquad \{X,f\} = X(f),$$
$$\{X,e_1\} = \nabla_X e_1, \qquad \{X,Y\} = [X,Y] - \Omega(X,Y).$$

for $f \in C^\infty(M)$, $e_1, e_2 \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$.

***Proposition* 4.13.** The relations above extend to a non-degenerate Poisson bracket of degree -2 on $\mathcal{M}^\nabla$. Hence, $\mathcal{M}^\nabla$ is a symplectic N-manifold of degree 2.

Proof.
Non-degeneracy of $\{\cdot,\cdot\}$ is immediate from non-degeneracy of $\langle\cdot,\cdot\rangle$, because for fixed $f \in C^\infty(M)$ $X(f) = 0$ $\forall X \in \Gamma(TM)$ implies $f \in \mathbb{R}$ and for fixed $X \in \Gamma(TM)$ $X(f) = 0$ $\forall f \in C^\infty(M)$ implies $X = 0$. Let us check that Leibniz's rule is satisfied:

$$\begin{aligned}
\{X, fg\} &= X(fg) = X(f)g + fX(g) = \{X,f\}g + f\{X,g\},\\
\{X, fe\} &= \nabla_X(fe) = X(f)e + f\nabla_X e = \{X,f\}e + f\{X,e\},\\
\{X, fY\} &= [X,fY] - \Omega(X,fY) = f[X,Y] + X(f)Y - f\Omega(X,Y) = f\{X,Y\} + \{X,f\}Y,\\
\{fX, g\} &= fX(g) = f\{X,g\} + \{f,g\}X,\\
\{fX, e\} &= \nabla_{fX} e = f\nabla_X e = f\{X,e\} + \{f,e\}X.
\end{aligned}$$

We note that these relations extend through Leibniz's rule as $\{\alpha, e\} = \alpha(\cdot,\dots,\cdot,e)$ and $\{X,\alpha\} = \nabla_X\alpha$ for $\alpha \in \Gamma(\Lambda^p E^*)$. Now $\{e_1,e_2\} = \{e_2,e_1\}$ and $\{X,Y\} = -\{Y,X\}$ are clear, so we proceed to prove the Jacobi identity. First,

$$\begin{aligned}
\{X, \{e_1,e_2\}\} &= X(\langle e_1,e_2\rangle) = \langle \nabla_X e_1, e_2\rangle + \langle e_1, \nabla_X e_2\rangle = \{\{X,e_1\},e_2\} + \{e_1,\{X,e_2\}\},\\
\{\{X,Y\},f\} &= [X,Y](f) = -Y(X(f)) + X(Y(f))) = \{\{X,f\},Y\} + \{X\{Y,f\}\},\\
\{\{X,Y\},e\} &= \nabla_{[X,Y]}e - \Omega(X,Y)(\cdot,e) = -\nabla_Y\nabla_X e + \nabla_X\nabla_Y e = \{\{X,e\},Y\} + \{X,\{Y,e\}\}
\end{aligned}$$

Finally, notice that $\{\{X,Y\},Z\} = [[X,Y],Z] + \nabla_Z\Omega(X,Y)$, so the Jacobi identity $\{\{X,Y\},Z\} = \{X,\{Y,Z\}\} + \{\{X,Z\},Y\}$ follows from the Jacobi identity for the Lie bracket of vector fields and from the second Bianchi identity for ∇. □

The above construction has the disadvantage of depending on a connection ∇. There is a construction of a **canonical** graded manifold $\mathcal{M}$ associated to the pseudo-Euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$ whose Poisson brackets are less explicit but which is useful because (when E has more structure; for example, that of a Courant algebroid) there exist invariants of E that can be expressed in terms of functions of $\mathcal{M}$ [21], [52], and it is desirable that we have a closed form for these invariants that does not depend on any connection. We proceed to describe this canonical construction.

Consider the graded manifold $T^*[2]E[1]$ with coordinates $(q^a, \xi^i, p_a, \rho_i)$. and consider also $T^*[2]E^*[1]$. It follows from Example 3.32 that these graded manifolds are canonically symplectic and it is easy to see that the **Legendre transformation** presented in Example 2.15 is a canonical symplectomorphism between them. The double vector bundle structure on $T^*[2]E[1]$ gives a canonical map $\pi : T^*[2]E[1] \to (E \oplus E^*)[1]$. Consider now the isometric embedding

$$\begin{aligned} i : E &\to E \oplus E^* \\ e &\mapsto e + \frac{1}{2}\langle e, \cdot \rangle, \end{aligned}$$

where the metric on $E \oplus E^*$ is the canonical pairing $\langle e_1 + \xi^1, e_2 + \xi^2 \rangle = \xi^1(e_2) + \xi^2(e_1)$. The graded manifold $\mathcal{M}$ that we are looking for is the one that completes the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "i_{\mathcal{M}}"] \arrow[d, "\pi_{\mathcal{M}}"] & T^*[2]E[1] \arrow[d, "\pi"] \\ E[1] \arrow[r, "i"] & (E \oplus E^*)[1]. \end{tikzcd}$$

So we may choose $\mathcal{M}$ as the pull-back of $T^*[2]E[1]$ along $i : E \to E \oplus E^*$. If the symplectic form on $T^*[2]E[1]$ is given in local coordinates by $dp_a dq^a + d\rho_i d\xi^i$, its pull-back to $\mathcal{M}$ is

$$\omega = dp_a dq^a - \frac{1}{2} g_{ij} d\xi^i d\xi^j,$$

where $\langle e_i, e_j \rangle = g_{ij}$ for a local basis $\{e_i\}_i$ of $\Gamma(E)$ with dual basis $\{\xi^i\}_i \in \Gamma(E^*) = \mathcal{A}^1$. This concludes the proof that $\mathcal{M}$ is a canonical symplectic graded manifold of degree 2 having $E[1]$ as its degree 1 part, as we wanted. For $U \subset M$ such that $E_{|U} \cong U \times V$ we have $\mathcal{M}_{|U} \cong T^*[2]U \times V[1]$, with infinitesimal transformations given by $H \in \mathcal{A}^2$, $H = v^a(q)p_a + M_{i,j}(q)\xi^i\xi^j$ as

$$\{H, q^b\} = v^b, \qquad \{H, \xi^k\} = M_{i,j}(q) g^{jk} \xi^i \qquad \{H, p_a\} = -\frac{\partial v^b}{\partial q^a} p_b - \frac{1}{2}\frac{\partial M_{i,j}}{\partial q^a}\xi^i\xi^j.$$

These correspond to changes of coordinates of the form

$$q^b = q^b(\tilde{q}), \qquad \xi^k = T^k_{\tilde{k}}(\tilde{q})\tilde{\xi}^k, \qquad p_a = \frac{\partial \tilde{q}^a}{\partial q^a}\tilde{p}_a + \frac{1}{2}\frac{\partial T^k_{\tilde{k}}}{\partial q^a} g_{kl} T^l_{\tilde{l}} \tilde{\xi}^k \tilde{\xi}^l. \tag{4.1}$$

with $T^k_{\tilde{k}} g_{kl} T^l_{\tilde{l}} = g_{\tilde{k}\tilde{l}}$ an orthogonal bundle transformation of E. The affine term in the expression for p_a means that, unlike in the previous construction with an affine connection, we cannot think of elements of $C^\infty(\mathcal{M})$ as sections of a vector bundle anymore. The only global description that we get is $C^\infty(\mathcal{M}) = \Gamma(\Lambda^2 E \otimes S^* \mathbb{A})/I$, as it was previously observed.

The graded manifolds $\mathcal{M}^\nabla$ and $\mathcal{M}$ are symplectomorphic, as we proceed to show. The connection ∇ determines a horizontal distribution on TE and hence a surjective submersion $\pi_\nabla : T^*[2]E[1] \to T^*[2]M$. So we have maps

$$\pi_\nabla \circ i_{\mathcal{M}} : \mathcal{M} \to T^*[2]M, \qquad \pi_{\mathcal{M}} : \mathcal{M} \to E[1]$$

and assembling these we obtain a map

$$\Xi_\nabla : \mathcal{M} \to \mathcal{M}^\nabla = T^*[2]M \oplus E[1]$$

by $\Xi_\nabla = (\pi_\nabla \circ i_{\mathcal{M}}) \oplus \pi_{\mathcal{M}}$ which we claim is a symplectomorphism of graded manifolds. Indeed, the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "i_{\mathcal{M}}"] \arrow[d, "\Xi_{\mathcal{M}}"] & T^*[2]E[1] \arrow[d, "\pi_\nabla \oplus \pi"] \\
T^*[2]M \oplus E[1] \arrow[r, "id \oplus i"] & T^*[2]M \oplus (E \oplus E^*)[1].
\end{tikzcd}$$

is commutative (in the first component it is immediate and in the second component it follows from the definition of $\mathcal{M}$) and the horizontal maps are symplectic embeddings, where we are using the symplectic structure from Example 3.34 on $T^*[2]M \oplus (E \oplus E^*)[1]$, so it suffices to show that $\pi_\nabla \oplus \pi$ is a symplectomorphism, and for this we just need to prove that the Poisson bracket is preserved for $f \in C^\infty(M)$, $e \in \Gamma(E)$, $\xi \in \Gamma(E^*)$ and $X \in \Gamma(TM)$. It is clear that $\pi^* f$ and $\pi^* e$ are constant functions in the fibers of $T^*[2]E[1]$ representing the same $f \in C^\infty(M)$ and $e \in \Gamma(E)$, while $\pi^*\xi$ and $\pi_\nabla^* X$ are fiberwise linear functions representing ι_ξ and ∇_X, by definition of π_∇ and π (through the Legendre transformation). Using the relations from Example 3.34 we see that the Poisson bracket coincides with the canonical one in $T^*[2]E[1]$.

A common way to work with these manifolds is to work on $\mathcal{M}^\nabla$ with a choice of connection ∇ and then use the map Ξ_∇ to obtain canonical results. For example, if $L \subset E$ is a subbundle, then there is a canonical embedding $i_\nabla : L[1] \to \mathcal{M}^\nabla = T^*[2]M \oplus E[1]$; hence, the composition $\Xi_\nabla^{-1} \circ i_\nabla : L[1] \to \mathcal{M}$ is a canonical embedding. In any case, we have proven:

***Proposition* 4.14.** The category of symplectic N-manifolds $(\mathcal{M}, \omega)$ such that $deg(\omega) = 2$ with morphisms of graded manifolds preserving the symplectic structure is equivalent to the opposite category of pseudo-Euclidean vector bundles $(E, \langle \cdot, \cdot \rangle)$ with orthogonal bundle morphisms. Under this identification, functions of degree 2 on $\mathcal{M}$ correspond to covariant differential operators of degree 1 (as differential operators) on E preserving $\langle \cdot, \cdot \rangle$.

***Remark* 4.15.** For $E \to M$ a pseudo-Euclidean vector bundle with corresponding symplectic N-manifold $\mathcal{M}$, let us study isotropic and Lagrangian submanifolds $i : \mathcal{N} \to \mathcal{M}$ having the same underlying manifold M. As in Remark 4.12, these are local properties, so we may think of $\mathcal{M}$ as $T^*[2]\mathbb{R}^n \oplus V[1]$ for $V \to \mathbb{R}^n$ a vector bundle with a non-degenerate pairing $\langle \cdot, \cdot \rangle$ and $T_p\mathcal{M}$ as $\mathbb{R}^n \oplus (\mathbb{R}^n)^* \oplus V_p \oplus V_p$, with ω restricting to the canonical pairing plus $\langle \cdot, \cdot \rangle$. Since we are imposing $\mathcal{N}$ to have $\mathbb{R}^n$ as underlying manifold, its tangent space at p is $\mathbb{R}^n \oplus W \oplus L_p \oplus L_p$ for $W \subset (\mathbb{R}^n)^*$ responsible for the degree 2 part of $\mathcal{N}$ and $L_p \subset V_p$. Then isotropy of $\mathcal{N}$ implies $W = \{0\}$ and in fact it is easy to see that

1. If $i : \mathcal{N} \to \mathcal{M}$ is an isotropic submanifold with M as underlying manifold, then $\mathcal{N} = L[1]$ for $L \subset E$ isotropic.

2. If $i : \mathcal{N} \to \mathcal{M}$ is a coisotropic submanifold with M as underlying manifold, then $\mathcal{N}_1 = L[1]$ for $L \subset E$ coisotropic (but $\mathcal{N}$ can have coordinates in degree 2).

3. If $i : \mathcal{N} \to \mathcal{M}$ is a Lagrangian submanifold with M as underlying manifold, then $\mathcal{N} = L[1]$ for $L \subset E$ a maximally isotropic subbundle.

4. If $i : \mathcal{N} \to \mathcal{M}$ is a symplectic submanifold with M as underlying manifold, then $\mathcal{N} = V[1]$ for $V \subset E$ such that $\langle \cdot, \cdot \rangle$ is non-degenerate on V.

4.3. Symplectic NQ-manifolds of Degree 1 and 2

In this section we conclude the proof of the Ševera-Roytenberg Theorem characterizing Courant algebroids as **symplectic NQ-manifolds** of degree 2. This will be done by studying the derived bracket induced by a Hamiltonian vector field on the sheaf of functions of the symplectic N-manifolds studied in Section 3.4. This study will also show that symplectic NQ-manifolds of degree 1 are in one-to-one correspondence with ordinary Poisson manifolds.

Theorem 4.16 (Roytenberg, [42]). *The category of symplectic NQ-manifolds $(\mathcal{M}, \omega, Q)$ such that $deg(\omega) = 1$ with Q-symplectomorphisms is equivalent to the category of ordinary Poisson manifolds.*

Proof.
Consider a symplectic NQ-manifold $(\mathcal{M}, \omega, Q)$ with $deg(\omega) = 1$. It follows from Proposition 4.11 that $\mathcal{M} \cong T^*[1]M$ for M the base manifold of $\mathcal{M}$. From Example 3.31, it follows that a Hamiltonia homological vector field Q on $\mathcal{M}$ is equivalent to a Poisson tensor on M. It is clear that for a map $F : T^*[1]M \to T^*[1]N$ arising from a map $f : M \to N$ preserves hamiltonian vector fields Q_M, Q_N if and only if it preserves the corersponding derived brackets; i.e., if and only if f is a Poisson map. □

It follows from Theorem 3.37 that each Poisson manifold (M, π) has a corresponding Lie algebroid structure on T^*M; this is sometimes called the **Poisson Lie algebroid** of (M, π).

***Remark* 4.17.** Recall Remark 4.12 and the definition of Q-isotropy (resp. Q-coisotropy, etc.) from Section 3.4. Let (M, π) be a Poisson manifold and consider a submanifold $E[1] \to T^*[1]M$ of the symplectic NQ-manifold $(T^*[1]M, \omega, \pi)$, with $E \to N$ a subbundle of j^*T^*M for a submanifold $j : N \to M$. By Theorem 3.37, $E[1]$ is a Q-manifold if and only if E is a Lie algebroid, and it is clear that $i : E[1] \to T^*[1]M$ is a Q-morphism if and only if the Lie algebroid structure on E is the restriction of the one in T^*M, which happens precisely when $(j^*\pi)(E) \subset TN$. In particular,

1. Q-Lagrangian submanifolds of $(T^*[1]M, \pi)$ are in bijection with submanifolds $N \to M$ such that $\pi(F_N) \subset TN$, for F_N the conormal bundle of N. These are called **coisotropic submanifolds** in ordinary Poisson geometry.

2. Q-symplectic submanifolds of $(T^*[1]M, \pi)$ are in bijection with **Poisson submanifolds $N \to M$**.

We are finally prepared for proving the result that motivated this whole work, which we present in Theorems 4.18 and 4.20 and in Corollary 4.21.

Theorem 4.18. *Let $(\mathcal{M}, \omega, Q)$ be a symplectic NQ-manifold, with $Q = \{\Theta, \cdot\}$ and let $(E, \langle\cdot,\cdot\rangle)$ be its corresponding pseudo-Euclidean vector bundle. Then, the relations*

$$a(e_1)f := \{\{S, e_1\}, f\} \qquad [e_1, e_2] := \{\{S, e_1\}, e_2\}$$

for $f \in C^\infty(M) = \mathcal{A}^0$ and $e_1, e_2 \in \Gamma(E) \cong_{\langle\cdot,\cdot\rangle} \Gamma(E^*) = \mathcal{A}^1$ define a structure of Courant algebroid on E.

Proof.
First notice that we can express the map d_E from Definition 4.1 as $d_E f = \{\Theta, f\}$, since $d_E f \in \Gamma(E)$ is determined by

$$\{d_E f, e\} = df(a(e)) = a(e)f = \{\{\Theta, e\}, f\} = \{\{\Theta, f\}, e\},$$

which implies $d_E f = \{\Theta, f\}$ by non-degeneracy. Now $\{S, \cdot\}$ is a differential on the Lie superalgebra $(C^\infty(\mathcal{M}), \{\cdot,\cdot\})$, which has $C^\infty(M) \oplus \Gamma(E) = \mathcal{A}^0 \oplus \mathcal{A}^1$ as a (non-abelian) subalgebra stable under the derived bracket induced by Θ. Then Proposition 3.2 gives us Properties 3 and 5 from Definition 4.1. Property 4 is an immediate consequence of Leibniz's rule for the Poisson bracket:

$$[e_1, fe_2] = \{\{S, e_1\}, fe_2\} = \{\{S, e_1\}, f\}e_2 + f\{\{S, e_1\}e_2\} = a(e_1)(f)e_2 + f[e_1, e_2],$$

while Property 2 follows from the Jacobi identity:

$$\begin{aligned} a(e_1)(\langle e_2, e_3\rangle) = \{\{S, e_1\}, \{e_2, e_3\}\} &= \{\{\{\{S, e_1\}, e_2\}, e_3\} + \{e_2, \{\{S, e_1\}, e_3\}\} \\ &= \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle. \end{aligned}$$

□

***Remark* 4.19.** By taking a look at the proof of Proposition 3.2 we see that $\{\Theta, \Theta\} = 0$ is only required for Property 3 of Courant algebroids. Thus, any $H \in \mathcal{A}^3$ induces in the same way as Θ an *almost Courant algebroid* structure on E which satisfies everything in Definition 4.1 except for the Jacobi identity for the Dorfman bracket.

Theorem 4.20. *Let E be a Courant algebroid and for a metric connection ∇ on E consider its corresponding graded symplectic manifold $\mathcal{M}^\nabla = (M, \Gamma(\Lambda^* E \otimes S^* TM))$. Then, there exists a unique $\Theta \in C_3^\infty(\mathcal{M}^\nabla)$ such that*

$$\{\{\Theta, e_1\}, f\} = a(e_1)(f), \quad \{\{\Theta, e_1\}, e_2\} = [e_1, e_2], \quad \{\Theta, f\} = d_E f. \tag{4.2}$$

It is given by $\Theta = a + T$, where $a \in \Gamma(E^* \otimes TM)$ is the anchor of E and $T \in \Gamma(\Lambda^3 E^*)$ is defined by $T(e_1, e_2, e_3) = \langle \nabla_{a(e_1)} e_2 - \nabla_{a(e_2)} e_1 - [e_1, e_2], e_3\rangle + \langle \nabla_{a(e_3)} e_1, e_2\rangle$. Moreover,

$$\{\Theta, \Theta\} = 0.$$

Proof.
Uniqueness follows directly from non-degeneracy of the Poisson bracket. We leave the proof that $T \in \Gamma(\Lambda^3 E^*)$ indeed for Proposition 4.27 below, where it is shown in a broader context. In order to obtain the identities (4.2) we choose a local frame $\{\xi^i\}_i$ of E with dual frame $\{\tilde{\xi}^i\}_i$ (that is, $\langle \xi^i, \tilde{\xi}^j\rangle = \delta_{i,j}$) so that we can write $a = \tilde{\xi}^i a(\xi^i)$. Then

$$\begin{aligned}\{\{a, e_1\}, f\} &= \{\tilde{\xi}^i(e_1)a(\xi^i), f\} + \{\tilde{\xi}^i \cdot \nabla_{a(\xi^i)} e_1, f\} = \{a(e_1), f\} = a(e_1)(f),\\ \{\{a, e_1\}, e_2\} &= \{\tilde{\xi}^i(e_1)a(\xi^i), e_2\} + \{\tilde{\xi}^i \cdot \nabla_{a(\xi^i)} e_1, e_2\} = \nabla_{a(e_1)} e_2 + \tilde{\xi}^i\langle \nabla_{a(\xi^i)} e_1, e_2\rangle - \nabla_{a(e_2)} e_1\\ &= T(e_1, e_2, \cdot) + [e_1, e_2];\end{aligned}$$

the second equation and the fact that $\{\{T, e_1\}, e_2\} = T(\cdot, e_2, e_1) = -T(e_1, e_2, \cdot)$ imply $\{\{\Theta, e_1\}, e_2\} = [e_1, e_2]$. Thus $\{\Theta, f\} \in \Gamma(E)$ is such that

$$\{\{\Theta, f\}, e_1\} = \{\{\Theta, e_1\}, f\} = a(e_1)(f),$$

which is precisely what $\{\Theta, f\} = d_E f$ means. Finally, these relations imply $\{\Theta, \Theta\} = 0$:

$$\begin{aligned}\{\{\{\Theta,\Theta\}, f\}, g\} &= \{\{\Theta, \{\Theta, f\}\}, g\} + \{\{\Theta, f\}, \Theta\}, g\}\\ &= \{\Theta, \{\{\Theta, f\}, g\}\} + 2\{\{\Theta, g\}, \{\Theta, f\}\} + \{\{\{\Theta, f\}, g\}, \Theta\}\\ &= 2\langle a^* dg, a^* df\rangle = 0,\\ \{\{\{\{\Theta,\Theta\}, e_1\}, e_2\}, f\} &= \{\{\{\Theta, \{\Theta, e_1\}\}, e_2\}, f\} - \{\{\{\{\Theta, e_1\}, \Theta\}, e_2\}, f\}\\ &= \{\{\Theta, \{\{\Theta, e_1\}, e_2\}\}, f\} + \{\{\{\Theta, e_2\}, \{\Theta, e_1\}\}, f\}\\ &\quad - \{\{\{\Theta, e_1\}, \{\Theta, e_2\}\}, f\} + \{\{\{\{\Theta, e_1\}, e_2\}, \Theta\}, f\}\\ &= 2a([e_1, e_2])(f) - 2[a(e_1), a(e_2)](f) = 0,\\ \{\{\{\{\Theta,\Theta\}, e_1\}, e_2\}, e_3\} &= \{\{\{\Theta, \{\Theta, e_1\}\}, e_2\}, e_3\} - \{\{\{\{\Theta, e_1\}, \Theta\}, e_2\}, e_3\}\\ &= \{\{\Theta, \{\{\Theta, e_1\}, e_2\}\}, e_3\} + \{\{\{\Theta, e_2\}, \{\Theta, e_1\}\}, e_3\}\\ &\quad - \{\{\{\Theta, e_1\}, \{\Theta, e_2\}\}, e_3\} + \{\{\{\{\Theta, e_1\}, e_2\}, \Theta\}, e_3\}\\ &= 2[[e_1, e_2], e_3] - 2\{\{\Theta, e_1\}, \{\{\Theta, e_2\}, e_3\}\} - 2\{\{\{\Theta, e_1\}, e_3\}, \{\Theta, e_2\}\}\\ &= 2[[e_1, e_2], e_3] - 2[e_1, [e_2, e_3]] - 2[e_2, [e_1, e_3]] = 0.\end{aligned}$$

and any $H \in C^\infty_4(\mathcal{M}^\nabla) = \Gamma(S^2TM \oplus TM \otimes \Lambda^2 E^* \oplus \Lambda^4 E^*)$ satisfying

$$\{\{H, f\}, g\} = 0, \qquad \{\{\{H, e_1\}, e_2\}, f\} = 0, \qquad \{\{\{\{H, e_1\}, e_2\}, e_3\}, e_4\} = 0$$

for any $f, g \in C^\infty(M)$ and $e_1, e_2, e_3, e_4 \in \Gamma(E)$ must be $H = 0$ by non-degeneracy. □

Composing with the map $\Xi_\nabla^{-1} : \mathcal{M}^\nabla \to \mathcal{M}$ gives the canonical form of the Hamiltonian $\Theta \in C^\infty(\mathcal{M})$. If $\{q^a, \xi^i, p_a\}$ are coordinates on $\mathcal{M}$ such that $\omega = dp_a dq^a - \frac{1}{2} g_{ij} d\xi^i d\xi^j$, then

$$\Theta = \xi^i a(e_i)(q^a) p_a - \frac{1}{6}\langle [e_i, e_j], e_k\rangle \xi^i \xi^j \xi^k$$

for $\{e_i\}_i$ the dual basis of $\{\xi^i\}_i$.

Corollary* 4.21 *(Roytenberg, [42]). The category of symplectic NQ-manifolds $(\mathcal{M}, \omega, Q)$ such that $deg(\omega) = 2$ is equivalent to the opposite category of Courant algebroids.

Proof.
Theorems 4.18 and 4.20 imply that there is a canonical bijection between both sets of objects. The discussion in Section 4.2 implies that morphisms of graded manifolds preserving the symplectic structure correspond to orthogonal maps of the corresponding pseudo-Euclidean bundles. These preserve the Courant algebroid structure if and only if they preserve the Hamiltonian Θ, by non-degeneracy of the Poisson bracket. $\square$

***Remark* 4.22.** For $E \to M$ a Courant algebroid with corresponding symplectic NQ-manifold $\mathcal{M}$, recall Remark 4.15. It is then clear that Q-isotropic submanifolds of $\mathcal{M}$ having M as underlying manifold are in one-to-one correspondence with isotropic subbundles of E that are closed under the Dorfman bracket. In particular, Lagrangian submanifolds having M as underlying manifold are almost Dirac structures and Q-Lagrangian submanifolds having M as underlying manifold are Dirac structures. A study of higher analogs of Dirac structures in terms of Q-Lagrangian submanifolds on symplectic NQ-manifolds has been carried out in [12].

4.4. Courant Algebroids & Graded Geometry

In this section we show how to interpret the structure of Courant algebroids from the perspective of graded geometry. We begin with a general remark on the relation between Courant algebroids and L_∞-algebras. Then we discuss two important examples in detail: **exact** Courant algebroids and **Lie bialgebroids**, as presented in Examples 4.7 and 4.8. In particular, we will show how the Q-cohomology of the graded manifold associated to a Courant algebroid encodes its deformations and we will characterize Dirac structures appearing as the graph of a skew-symmetric tensor on the double of a Lie bialgebroid.

***Remark* 4.23.** As we mentioned at the ending of Section 3.5, NQ-manifolds are related to L_∞-algebroids. For $\mathcal{M}$ the symplectic NQ-manifold corresponding to a Courant algebroid $(E, a, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$, we saw in Remark 4.2 that there is an L_∞-structure on $V := C^\infty(M)[1] \oplus \Gamma(E)$ which we can now describe as the projection onto $V[-1] \subset C^\infty(\mathcal{M})$ of the following higher derived brackets determined by Θ:

$$\begin{aligned} l_1(F_1) &= \{\Theta, F_1\}, \\ l_2(F_1, F_2) &= \frac{1}{2}\operatorname*{cycl}_{1,2}\{\{\Theta, F_1\}, F_2\}, \\ l_3(F_1, F_2, F_3) &= -\frac{1}{6}\operatorname*{cycl}_{1,2,3}\{\{\{\Theta, F_1\}, F_2\}, F_3\}, \end{aligned}$$

for $F_1, F_2, F_3 \in V[-1]$. This is the same way in which the brackets of a finite-dimensional L_∞-algebra L are constructed from a homological vector field on $(\{*\}, S^*(L[1])^*)$, since we can identify each section $e \in \Gamma(E)$ with its Hamiltonian vector field ι_e and then $\{\Theta, e\} = [Q, \iota_e]$ for $Q = \{\Theta, \cdot\}$.

Example* 4.24 *(Exact Courant algebroids Revisited). Given an ordinary manifold M, we consider the symplectic N-manifold $\mathcal{M} = T^*[2]T[1]M$ (see Example 3.32) with local coordinates $\{q^a, \xi^a, p_a, \theta_a\}$, $deg(q^a) = 0$, $deg(\xi^a) = deg(\theta_a) = 1$, $deg(p_a) = 2$ and its canonical symplectic structure $\omega = dp_a dq^a + d\theta_a d\xi^a$. There is an isomorphism of graded manifolds $T^*[2]T[1]M \cong T[1]T^*[1]M$; if $\{q^a, \rho_a, \eta^a, \nu_a\}$, $deg(q^a) = 0$, $deg(\rho_a) = deg(\eta^a) = 1$,

$deg(v_a) = 2$ are coordinates on $T[1]T^*[1]M$, the isomorphism is simply $q^a \mapsto q^a$, $\xi^a \mapsto \eta^a$, $p_a \mapsto v_a$, $\theta_a \mapsto \rho_a$. Thus, there is a canonical choice for a homological vector field on $T^*[2]T[1]M$, which is the de Rahm differential (or, more precisely, the pull-back of the de Rahm differential from $T[1]T^*[1]M$, see Example 3.15)

$$Q = d_{dR} = \xi^a \frac{\partial}{\partial q^a} + p_a \frac{\partial}{\partial \theta_a}.$$

Moreover, this is a Hamiltonian vector field and its Hamiltonian function is $\Theta = p_a \xi^a \in \mathcal{A}^{1,2}$. In fact, d_{dR} can also be interpreted in a canonical way as the Hamiltonian lift of the exterior derivative $d \in Der\, C^\infty(T[1]M)$ to $T^*[2]T[1]M$, as in Example 3.32. Treating $\mathcal{M}$ as $T^*[2]T[1]M$ we see that $\Omega(M) = C^\infty(T[1]M) \subset C^\infty(\mathcal{M})$ and treating it as $T[1]T^*[1]M$ we see that $\Gamma(\Lambda^* TM) = C^\infty(T^*[1]M) \subset C^\infty(\mathcal{M})$. Because d_{dR} is the Hamiltonian lift of the exterior derivative, for $\alpha \in \Omega(M)$ we have $d_{dR}\alpha = d\alpha$ in the usual sense. On the other hand, for $X = \sum f^b \theta_b \in \Gamma(TM)$,

$$\{p_a \xi^a, f^b \theta_b\} = p_a \{\xi^a, \theta_b\} f^b + \xi^a \{p_a, f^b\} \theta_b = f^a p_a + \frac{\partial f^b}{\partial q^a} \xi^a \theta_b,$$

so we obtain the anchor

$$a(X+\alpha) f = \{\{\Theta, X+\alpha\}, f\} = \{f^a p_a + \frac{\partial f^b}{\partial q^a} \xi^a \theta_b + d\alpha, f\} = f^a \{p_a, f\} = X(f) \quad f \in C^\infty(M)$$

and the Dorfman bracket

$$\begin{aligned}
[X+\alpha, Y+\beta] &= \{\{\Theta, X+\alpha\}, Y+\beta\} = \{f^a p_a + \frac{\partial f^b}{\partial q^a} \xi^a \theta_b + d\alpha, Y+\beta\} \\
&= f^a \{p_a, Y\} - \frac{\partial f^b}{\partial q^a} \theta_b \{\xi^a, Y\} + f^a \{p_a, \beta\} + \frac{\partial f^b}{\partial q^a} \xi^a \{\theta_b, \beta\} + \{d\alpha, Y\} \\
&= [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha.
\end{aligned}$$

Let us study which other Courant algebroid structures with the same anchor and metric are there on $TM \oplus T^*M$. Any such structure will be given by $\{\Theta + H, \cdot\}$ for some $H \in C_3^\infty(\mathcal{M})$ satisfying $\{\Theta, H\} = \{H, H\} = \{\{H, X+\alpha\}, f\} = 0$ for all $X + \alpha \in \Gamma(TM \oplus T^*M)$ and $f \in C^\infty(M)$. In particular, H determines a degree 3 cohomology class on the Q-cohomology of $\mathcal{M}$. Given H_1, H_2 like those, if $\exists G \in C_2^\infty(\mathcal{M})$ such that $H_1 - H_2 = \{\Theta, G\}$ then $\{G, \cdot\}$ is a symplectic vector field preserving d_{dR}; if it can be integrated to a symplectic diffeomorphism it will relate the Courant algebroid structures arising from H_1 and H_2. In other words, the third Q-cohomology group of $\mathcal{M}$ represents the infinitesimal deformations of the Courant algebroid structure.

In order to describe this space in a more precise way, we first notice that the Legendre transformation studied in Section 4.2 induces a $\mathbb{Z} \times \mathbb{Z}$-grading on $\mathcal{M}$, described by the Euler vector fields (see 2.23)

$$E_1 = p_a \frac{\partial}{\partial p_a} + \theta^a \frac{\partial}{\partial \theta^a}, \qquad E_2 = p_a \frac{\partial}{\partial p_a} + \xi^a \frac{\partial}{\partial \xi^a},$$

and our original grading is $E = E_1 + E_2$. We write $\mathcal{A}^{p,q}$ for functions of degree p with respect to E_1 and degree q with respect to E_2. Notice $\mathcal{A}^{0,\cdot} = \Omega(M)$, while $\mathcal{A}^{1,\cdot}$ are derivations of $\Omega(M)$ and, in general, the space $\mathcal{A}^{p,\cdot}$ can be thought of as the space of symbols of differential operators of order p on $T[1]M$. Interestingly, d_{dR} has weight 0 with respect to the grading E_1, so d_{dR} is, for each $r \geq 0$, a differential on the complex $\mathcal{A}^{r,\cdot}$. We claim that all the complexes with $r \geq 1$ are acyclic. To prove this claim, it suffices to show that there exists $\iota \in Der\, C^\infty(\mathcal{M})$ such that $[Q, \iota] = E_1$ (that is, E_1 is a coboundary for $[Q, \cdot]$) because in that case, for $f \in \mathcal{A}^{r,\cdot}$ $r \geq 1$ with $Q(f) = 0$, we will have $f = E_1(f/r) = Q\,\iota(f/r)$. It is an easy check that $\iota = \theta_i \frac{\partial}{\partial p_i}$ does the deal.

This means that the whole Q-cohomology of this Courant algebroid is given by the complex $(\mathcal{A}^{0,\cdot}, Q)$, which is nothing but the ordinary de Rahm complex of M. We proceed to describe the way differential forms act on $\mathcal{M}$. If we have a 1-form $\sigma \in \Omega^1(M) \subset \mathcal{A}^1$ we see that $[\sigma, Y + \beta] = -\iota_Y d\sigma$, which determines an action on $TM \oplus T^*M$ given by the bundle morphism

$$\begin{aligned} \varphi_\sigma : TM \oplus T^*M &\to TM \oplus T^*M \\ Y + \beta &\mapsto Y + \beta + \iota_Y d\sigma \end{aligned}$$

which is trivial if σ is closed. For a 2-form $\tau \in \Omega^2(M) \subset \mathcal{A}^2$, we have $\{\tau, Y + \beta\} = -\iota_Y \tau$ and we can similarly define the automorphism

$$\begin{aligned} \varphi_\tau : TM \oplus T^*M &\to TM \oplus T^*M \\ Y + \beta &\mapsto Y + \beta + \iota_Y \tau \end{aligned}$$

which always preserves $\langle \cdot, \cdot \rangle$ and the anchor, but

$$[X+\alpha+\iota_X\tau, Y+\beta+\iota_Y\tau] = [X+\alpha, Y+\beta]+\mathcal{L}_X\iota_Y\tau-\iota_Y d\iota_X\tau = [X+\alpha, Y+\beta]+\iota_{[X,Y]}\tau+\iota_Y\iota_X d\tau,$$

which means that $[\cdot, \cdot]$ is preserved precisely when $d\tau = 0$. If $\tau = d\sigma$, then $\varphi_\sigma = \varphi_\tau$. Finally, for a three-form $\eta \in \Omega^3(M)$ we know that $\Theta + \eta$ is a Hamiltonian function that induces a homological vector field if and only if $\{\Theta, \eta\} = d\eta = 0$ ($\{\eta, \eta\} = 0$ for every $\eta \in \Omega^3(M)$ since the Poisson bracket is an extension of the metric on $TM \oplus T^*M$). In this case, since $\{\{\eta, X + \alpha\}, Y + \beta\} = \iota_Y \iota_X \eta$, we will obtain a new bracket given by

$$[X + \alpha, Y + \beta]_\eta = [X, Y] + \mathcal{L}_X \beta - \iota_Y \alpha + \iota_Y \iota_X \eta.$$

If $\eta = d\tau$, the Courant algebroid structure that we obtain is isomorphic to the canonical one, meaning that the vector bundle automorphism φ_τ defined above preserves $\langle \cdot, \cdot \rangle$ and the anchor, and it relates both brackets. This proves the claims that we made in Example 4.7.

Example 4.24 shows the kind of information presented in the Q-cohomology groups of the graded manifold $\mathcal{M}$ associated to a Courant algebroid E. A systematic treatment is the following: For $f \in \mathcal{A}^0 = C^\infty(M)$, we have already seen that $\{\Theta, f\} = d_E f \in \Gamma(E)$; that is,

$$\{\Theta, f\} = 0 \qquad \Leftrightarrow \qquad \langle Df, e \rangle = df(a(e)) = 0 \quad \forall e \in \Gamma(E).$$

So $H^0_\Theta(\mathcal{M})$ is the space of functions on M that are constant along the image of the anchor. Now for $e \in \mathcal{A}^1 = \Gamma(E)$ we see that $\{\Theta, e\} \in \mathcal{A}^2$ is the Hamiltonian of a degree 0 symplectic vector field; that is, a CDO on E preserving $\langle\cdot,\cdot\rangle$. Then we see that $H^1_\Theta(\mathcal{M})$ is the space of sections $e \in \Gamma(E)$ such that their corresponding action is trivial modulo those of the form $d_E f$ for $f \in C^\infty(M)$. Now elements $D \in \mathcal{A}^2$ are CDO's on E preserving $\langle\cdot,\cdot\rangle$ which act as $\{D,\cdot\}$, and they preserve the Dorfman bracket (that is, $\{\Theta,\cdot\}$) precisely when $\{\Theta, D\} = 0$, so $H^2_\Theta(\mathcal{M})$ gives structure-preserving infinitesimal transformations modulo those of the form $\{\Theta, e\} = [e,\cdot]$ for $e \in \Gamma(E)$. For any $H \in \mathcal{A}^3$ we see that $\{\Theta, H\} = 0 \Leftrightarrow \{\Theta + tH, \Theta + tH\} = O(t^2)$, so $H^3_\Theta(\mathcal{M})$ is the space of infinitesimal deformations of the Courant algebroid structure modulo the trivial ones that appear as $\{\Theta, D\}$, for $D \in \mathcal{A}^2$. Each of these $H \in \mathcal{A}^3$ gives a new Courant algebroid structure if and only if $\{H, H\} = 0$; otherwise, $\{H, H\}$ defines a non-trivial cohomology class on $H^4_\Theta(\mathcal{M})$.

This discussion shows that the analog of a Ševera class for non-exact Courant algebroids is the third Q-cohomology group of $\mathcal{M}$, at least at the infinitesimal level. In fact, this is true for any symplectic NQ-manifold $\mathcal{M}$ of degree $d \geq 1$: It follows from Proposition 3.24 that all symplectic vector fields of degree 0 are given by Hamiltonian functions of degree d. That is, the Lie algebra of the symplectomorphism group of $\mathcal{M}$ is $\mathcal{A}^d$. These Hamiltonians preserve the Q-structure precisely when they define a cohomology class, so $H^d_Q(\mathcal{M})$ represents infinitesimal symplectomorphisms preserving Q modulo the trivial ones that appear as $Q(f)$ for $f \in C^\infty_{d-1}(\mathcal{M})$ and, as before, $H^{d+1}_Q(\mathcal{M})$ determines infinitesimal deformations of the Q-structure on $\mathcal{M}$.

Example* 4.25 *(The Double of a Lie Bialgebroid Revisited). Consider two Lie algebroids $(A, a_A, [\cdot,\cdot]_A)$ and $(A^*, a_{A^*}, [\cdot,\cdot]_{A^*})$ which are in duality as vector bundles. Then it follows from Vaintrob's Theorem 3.37 that we have Q-manifolds $(A[1], d_A)$ and $(A^*[1], d_{A^*})$. Moreover, it follows from Example 3.32 that these homological vector fields lift in a Hamiltonian way to the cotangent spaces $T^*[2]A[1]$ and $T^*[2]A^*[1]$; call their Hamiltonian functions Θ_A and Θ_{A^*}. Again, the Legendre transformation from Section 4.2 shows that these two symplectic graded manifolds of degree 2 are canonically isomorphic, so we may regard $\{\Theta_A,\cdot\}$ and $\{\Theta_{A^*},\cdot\}$ as two different homological vector fields on a single graded manifold $\mathcal{M}$. If $\{q^a, \xi^i, p_a, e_i\}$ are local coordinates, then

$$\Theta_A = -\frac{1}{2}\sum_{i,j,k} c^k_{i,j}\xi^i\xi^j e_k + \sum_{i,a} a_A(e_i)(q^a)\xi^i p_a, \qquad \Theta_{A^*} = -\frac{1}{2}\sum_{i,j,k} \tilde{c}^{i,j}_k e_i e_j \xi^k + \sum_{i,a} a_{A^*}(\xi^i)(q^a) e_i p_a$$

for $[e_i, e_j]_A = c^k_{i,j} e_k$ and $[\xi_i, \xi_j]_{A^*} = \tilde{c}^{i,j}_k \xi^k$. As before, each fibration $\mathcal{M} \to A[1]$ and $\mathcal{M} \to A^*[1]$ has an associated grading w_A, w_{A^*} such that the original grading is $w_A + w_{A^*}$; if we write $\mathcal{A}^{p,q}$ for functions of degree p with respect to w_A and degree q with respect to w_{A^*}, then

$$\Theta_A \in \mathcal{A}^{1,2}, \qquad \Theta_{A^*} \in \mathcal{A}^{2,1},$$

and the Poisson bracket has bidegree $(-1,-1)$. It has also been shown in Section 3.4 that the canonical pseudo-Euclidean vector bundle corresponding to $\mathcal{M}$ is $A \oplus A^*$ with its obvious pairing, so it follows from Theorem 4.18 that we have two Courant algebroid structures on $E = A \oplus A^*$. The same computations as in Example 4.24 show that the Courant anchor a_{Θ_A}

and the Dorfman bracket $[\cdot,\cdot]_{\Theta_A}$ determined by Θ_A are, for $v_i + \xi_i \in \Gamma(A \oplus A^*)$,

$$\begin{aligned} a_{\Theta_A}(v+\xi) &= a_A(v), \\ [v_1+\xi_1, v_2+\xi_2]_{\Theta_A} &= [v_1,v_2]_A + \mathcal{L}^A_{v_1}\xi_2 - \iota_{v_2} d_A \xi_1, \end{aligned}$$

where $\mathcal{L}_A$ is the Lie derivative determined by d_A as $\mathcal{L}^A_v \xi = \iota_v d_A \xi + d_A \iota_v \xi$. Moreover, $d_A = \{\Theta_A, \cdot\}$ on $\Gamma(\Lambda^* A)$ and the odd Poisson bracket $[\cdot,\cdot]_A$ is the derived bracket $[\alpha,\beta]_A = (-1)^{p+1}\{\{\Theta_A,\alpha\},\beta\}$ for $\alpha \in \Gamma(\Lambda^p A)$ and $\beta \in \Gamma(\Lambda^q A)$. Of course, similar formulas hold for the Courant algebroid structure determined by Θ_{A^*}. Notice that the converse is also true: given any $\tilde{\Theta}_A \in \mathcal{A}^{1,2}$ (resp. $\tilde{\Theta}_{A^*} \in \mathcal{A}^{2,1}$) such that $\{\tilde{\Theta}_A, \tilde{\Theta}_A\} = 0$ (resp. $\{\tilde{\Theta}_{A^*}, \tilde{\Theta}_{A^*}\} = 0$), then $\{\tilde{\Theta}_A, \cdot\}$ (resp. $\{\tilde{\Theta}_{A^*}, \cdot\}$) projects to a homological vector field on $A[1]$ (resp. $A^*[1]$) and so it induces a Lie algebroid structure on A (resp. $A^*[1]$).

Now $\Theta := \Theta_A + \Theta_{A^*}$ is a function of degree 3 on $\mathcal{M}$ which commutes with itself (and thus induces a new Courant algebroid structure) if and only if $\{\Theta_A, \Theta_{A^*}\} = 0$; in this case we will obtain the Courant algebroid structure from Example 4.8. We claim that the condition $\{\Theta_A, \Theta_{A^*}\} = 0$ is equivalent to (A, A^*) being a Lie bialgebroid. To see this, we first note that $H := \{\Theta_A, \Theta_{A^*}\}$ has bidegree $(2,2)$. This means that $H = 0$ if and only if, for all $f, g \in C^\infty(M)$, $e_1, e_2 \in \Gamma(A)$ and $\xi_1, \xi_2 \in \Gamma(A^*)$,

$$\{\{H,f\},g\} = 0, \qquad \{\{\{\{H,\xi_1\},\xi_2\},e_1\},e_2\} = 0, \qquad \{\{\{\{H,e_1\},e_2\},\xi_1\},\xi_2\} = 0.$$

In fact, $\{\{H,f\},g\} = 0, \forall f,g \in C^\infty(M)$ implies $H \in \Lambda^2\Gamma(A)\otimes\Lambda^2\Gamma(A^*)$ and so in this case the latter two conditions above are equivalent. Notice then that, for $\gamma \in \Gamma(\Lambda^p A^*) \subset C^\infty(\mathcal{M})$ and $\eta \in \Gamma(\Lambda^q A^*) \subset C^\infty(\mathcal{M})$,

$$\begin{aligned} [d_A\gamma,\eta]_{A^*} + (-1)^{p+1}[\gamma, d_A\eta]_{A^*} &= (-1)^p\{\{\Theta_{A^*},\{\Theta_A,\gamma\}\},\eta\} + \{\{\Theta_{A^*},\gamma\},\{\Theta_A,\eta\}\} \\ &= (-1)^p\{\{\{\Theta_{A^*},\Theta_A\},\gamma\},\eta\} + (-1)^{p+1}\{\{\Theta_A,\{\Theta_{A^*},\gamma\}\},\eta\} \\ &\quad + \{\{\{\Theta_{A^*},\gamma\},\Theta_A\},\eta\} + (-1)^{p+1}\{\Theta_A,\{\{\Theta_{A^*},\gamma\},\eta\}\} \\ &= \{\{\{\Theta_{A^*},\Theta_A\},\gamma\},\eta\} + d_A[\gamma,\eta]_{A^*}, \end{aligned}$$

which shows that (A, A^*) is a Lie bialgebroid if and only if $\{\Theta_A, \Theta_{A^*}\} = 0$. In particular, (A, A^*) is a Lie bialgebroid if and only if so is (A^*, A). In conclusion, Courant algebroid structures on $T^*[2]A[1]$ given by some $\tilde{\Theta} = \tilde{\Theta}_A + \tilde{\Theta}_{A^*} \in \mathcal{A}^{1,2} \oplus \mathcal{A}^{2,1}$ with $\{\tilde{\Theta}_A, \tilde{\Theta}_A\} = \{\tilde{\Theta}_A, \tilde{\Theta}_{A^*}\} = \{\tilde{\Theta}_{A^*}, \tilde{\Theta}_{A^*}\} = 0$ are in bijection with Lie bialgebroid structures (A, A^*).

This framework also shows in a clean way how to twist this Courant algebroid structure: infinitesimal deformations of the Courant algebroid structure given by Θ are parameterized by the third $d_A + d_{A^*}$−cohomology group of $\mathcal{M}$. For the same reasons as in Example 4.24, d_A is a differential on each complex $\mathcal{A}^{p,\cdot}$ and d_{A^*} is a differential on each complex $\mathcal{A}^{\cdot,q}$. Thus, for example, $H \in \mathcal{A}^{3,0} \oplus \mathcal{A}^{0,3}$ is $d_A + d_{A^*}$-closed if and only if it is both d_A− and d_{A^*}−closed, and in this case $\Theta + H$ will give a Courant algebroid structure if and only if $\{H^{3,0}, H^{0,3}\} = 0$. Moreover, in this case H is $d_A + d_{A^*}$-exact if and only if it is both d_A− and d_{A^*}−exact. However, for general $H \in \mathcal{A}^3$, the interplay between d_A and d_{A^*} must be taken into account.

We finish this example by noting an interesting fact about Dirac structures on $A \oplus A^*$ which was first proved in [37] without graded geometry. For each $S \in \Gamma(\Lambda^2 A^*)$, there is a corresponding almost Dirac structure given by $L_S = \{e + \{S, e\} \ : \ e \in \Gamma(A)\}$ (notice $\{S, e\} \in \Gamma(A^*)$) because this is isotropic:

$$\{e_1 + \{S, e_1\}, e_2 + \{S, e_2\}\} = \{e_1, \{S, e_2\}\} + \{\{S, e_1\}, e_2\} = \{S, \{e_1, e_2\}\} = 0$$

and $r := rank(A) = rank(L_S)$, so L_S is maximal because the metric on $A \oplus A^*$ has signature (r, r). Let us study when L_S is in fact a Dirac structure. We can compute the Dorfman bracket with Θ:

$$\begin{aligned}
\{\{\Theta, e_1 + \{S, e_1\}\}, e_2 + \{S, e_2\}\}
&= \{\{\Theta, e_1\}, e_2\} + \{\{\Theta, e_1\}, \{S, e_2\}\} + \{\{\Theta, \{S, e_1\}\}, e_2 + \{S, e_2\}\} \\
&= \{\{\Theta, e_1\}, e_2\} + \{S, \{\{\Theta, e_1\}, e_2\}\} + \{\{\Theta, e_1\}, S\}, e_2\} - \{\{\Theta, \{e_1, S\}\}, e_2 + \{S, e_2\}\} \\
&= \{\{\Theta, e_1\}, e_2\} + \{S, \{\{\Theta, e_1\}, e_2\}\} + \{\{\Theta, S\}, e_1\}, e_2\} + \{\{\Theta, \{S, e_1\}\}, \{S, e_2\}\}.
\end{aligned}$$

By studying the bidegree of each of these terms we see that the above expression belongs to L_S if and only if

$$\{S, \{\{\{\Theta_{A^*}, S\}, e_1, \}, e_2\}\} = \{\{\{\Theta_A, S\}, e_1\}, e_2\} + \{\{\Theta_{A^*}, \{S, e_1\}\}, \{S, e_2\}\}$$

and we can compute

$$\begin{aligned}
\{\{\Theta_{A^*}, \{S, e_1\}\}, \{S, e_2\}\} &= \{\{\{\Theta_{A^*}, \{S, e_1\}\}, S\}, e_2\} + \{S, \{\{\Theta_{A^*}, \{S, e_1\}\}, e_2\}\} \\
&= \{\{\{\Theta_{A^*}, S\}, \{S, e_1\}\}, e_2\} + \{\{\Theta_{A^*}, \{\{S, e_1\}, S\}\}, e_2\} + \{S, \{\{\Theta_{A^*}, \{S, e_1\}\}, e_2\}\} \\
&= \{\{\{\{\Theta_{A^*}, S\}, S\}, e_1\}, e_2\} + \{\{S, \{\{\Theta_{A^*}, S\}, e_1\}\}, e_2\} \\
&\quad + \{\{\Theta_{A^*}, \{\{S, e_1\}, S\}\}, e_2\} + \{S, \{\{\Theta_{A^*}, \{S, e_1\}\}, e_2\}\} \\
&= \{\{\{\{\Theta_{A^*}, S\}, S\}, e_1\}, e_2\} - \{\{\Theta_{A^*}, \{S, e_1\}\}, \{S, e_2\}\} \\
&\quad + \{\{\Theta_{A^*}, \{\{S, e_1\}, S\}\}, e_2\} + 2\{S, \{\{\Theta_{A^*}, \{S, e_1\}\}, e_2\}\}.
\end{aligned}$$

Now $\{\{S, e_1\}, S\} = 0$, so this relation can be written as

$$\{\{\Theta_{A^*}, \{S, e_1\}\}, \{S, e_2\}\} = \frac{1}{2}\{\{\{\{\Theta_{A^*}, S\}, S\}, e_1\}, e_2\} + \{S, \{\{\Theta_{A^*}, \{S, e_1\}\}, e_2\}\}.$$

Substituting above we obtain that L_S is a Dirac structure if and only if

$$\{\Theta_A, S\} + \frac{1}{2}\{\{\Theta_{A^*}, S\}, S\} = 0.$$

The way to interpet this equation is the following: $\Gamma(\Lambda^* A^*)$ is an abelian subalgebra of the differential Lie superalgebra $(C^\infty(M), \{\cdot,\cdot\}, \{\Theta_{A^*}, \cdot\})$; hence, by Corollary 3.3, $(\Gamma(\Lambda A^*), [\cdot,\cdot]_{\Theta_{A^*}})$ is a Lie superalgebra, where $[\cdot,\cdot]_{\Theta_{A^*}}$ is the derived bracket induced by Θ_{A^*}. Moreover, since (A, A^*) is a Lie bialgebroid, $(\Gamma(\Lambda A^*), [\cdot,\cdot]_{\Theta_{A^*}}, \{\Theta_A, \cdot\})$ is a **differential** Lie superalgebra and the above equation is its Maurer Cartan equation. When $A = TM$ with its obvious Lie algebroid structure and $A^* = T^*M$ with zero anchor and bracket, this equation reads $d\omega = 0$ for $\omega \in \Gamma(\Lambda^2 T^*M)$ and if we invert the roles of TM and T^*M it reads $[\pi, \pi] = 0$ for $\pi \in \Gamma(\Lambda^2 TM)$.

4.5. Generalized Riemannian Geometry

In this section we introduce the main tools of **generalized Riemannian geometry**. As explained in Section 1.1, two-dimensional σ-models show that a full geometric understanding of the bundle $TM \oplus T^*M$ is helpful in physics. Generalized Riemannian geometry is the study of analogs of (pseudo)-Riemannian metrics, connections and curvature for Courant algebroids, such as $TM \oplus T^*M$. It was initiated by Hitchin in [26], who wanted to characterize special geometry in low dimensions by means of invariant polynomials on differential forms.

Definition 4.26. *A **generalized connection** D on a Courant algebroid E is a map $D : \Gamma(E) \to \Gamma(E^* \otimes E)$ such that, for $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$,*

$$\begin{aligned} a(e_1)\langle e_2, e_3\rangle &= \langle D_{e_1} e_2, e_3\rangle + \langle e_2, D_{e_1} e_3\rangle, \\ D_{e_1}(f e_2) &= a(e_1) f \cdot e_2 + f D_{e_1}(e_2), \end{aligned} \tag{4.3}$$

*where $D_e := \iota_e D$. The **torsion** of D is*

$$T(e_1, e_2, e_3) := \langle D_{e_1} e_2 - D_{e_2} e_1 - [e_1, e_2], e_3\rangle + \langle D_{e_3} e_1, e_2\rangle,$$

*its **curvature** is*

$$\Omega_{e_1, e_2}(e_3, e_4) := \langle D_{e_1} D_{e_2} e_3 - D_{e_2} D_{e_1} e_3 - D_{[e_1, e_2]} e_3, e_4\rangle.$$

*and its **divergence** is*

$$div(e) := Tr\, D(e),$$

where $Tr\, D(e)$ denotes the trace of the operator $s \mapsto D_s(e)$.

Note that the Dorfman bracket is not involved in the definition of generalized connections and so the same D is a generalized connection for different structures of Courant algebroid with the same anchor and pairing. Generalized connections always exist. Namely, if $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a metric connection, then $D_e = \nabla_{a(e)}$ always defines a generalized connection. Any two generalized connections D, D' are related by $D - D' = \chi$, for $\chi \in \Gamma(E^* \otimes \mathfrak{o}(E))$. For a fixed $e \in \Gamma(E)$, it is also convenient to define the **covariant derivative** of any $\alpha \in \Gamma(\Lambda^p E^*)$ with respect to e as

$$D_e \alpha(e_1, \dots, e_p) := a(e)(\alpha(e_1, \dots, e_p)) - \frac{1}{(p-1)!} \operatorname*{cycl}_{1,\dots,p} \alpha(D_e e_1, e_2, \dots, e_p);$$

that is, D_e is the unique derivation of $\Gamma(\Lambda^* E)$ extending the action of D_e on $\Gamma(E)$ and such that $D_e(f) = a(e)(f)$ for $f \in C^\infty(M)$. Here we are using the following notation for sums over a set of permutations which will be useful throughout the rest of the chapter:

$$\operatorname*{cycl}_{1,\dots,n} A(1, \dots, n) := \sum_{\sigma \in S_n} (-1)^\gamma A(\sigma(1), \dots, \sigma(n)),$$

where S_n is the set of permutations of $\{1, \dots, n\}$ and $(-1)^\gamma = sgn(\sigma)$ is the signature of the permutation σ.

***Proposition* 4.27.** The following properties are satisfied by the torsion, curvature and divergence of a generalized connection D.

1. $T \in \Gamma(\Lambda^3 E^*)$.
2. For fixed e_1, e_2, $\Omega_{e_1,e_2} \in \Lambda^2\Gamma(E^*)$.
3. For fixed e_3, e_4, $\Omega(e_3, e_4)$ satisfies

$$
\begin{aligned}
\Omega_{e_1,e_2}(e_3, e_4) &= -\Omega_{e_2,e_1}(e_3, e_4) - \langle D_{d_E\langle e_1,e_2\rangle} e_3, e_4\rangle,\\
\Omega_{e_1,fe_2}(e_3, e_4) &= f\Omega_{e_1,e_2}(e_3, e_4),\\
\Omega_{fe_1,e_2}(e_3, e_4) &= f\Omega_{e_1,e_2}(e_3, e_4) - \langle e_1, e_2\rangle\langle D_{d_E f} e_3, e_4\rangle.
\end{aligned}
\tag{4.4}
$$

4. $div(fe) = a(e)(f) + f div(e)$.

Proof.
First, $T(e_1, e_2, e_3)$ is clearly $C^\infty(M)$-linear on e_3. For e_1, e_2, it follows easily from $[e_1, fe_2] = f[e_1, e_2] + a(e_1)(f)e_2$ and $D_{e_1}(fe_2) = fD_{e_1}e_2 + a(e_1)(f)e_2$. Skew-symmetry on e_1, e_2 follows from $[e_1, e_2] + [e_2, e_1] = d_E\langle e_1, e_2\rangle$. Now

$$T(e_1, e_2, e_3) + T(e_1, e_3, e_2) = \langle D_{e_1}e_2 - [e_1, e_2], e_3\rangle + \langle D_{e_1}e_3 - [e_1, e_3], e_2\rangle = 0,$$

where the last step follows from (4.3) and 2 in Definition 4.1. Now Ω_{e_1,e_2} is clearly $C^\infty(M)$-linear on e_4 and so is on e_3 because of (4.3) and $[a(e_1), a(e_2)] = a([e_1, e_2])$:

$$
\begin{aligned}
\Omega_{e_1,e_2}(fe_3, e_4) &= \langle D_{e_1}D_{e_2}(fe_3) - D_{e_2}D_{e_1}(fe_3) - D_{[e_1,e_2]}(fe_3), e_4\rangle\\
&= \langle D_{e_1}a(e_2)(f)e_3 - D_{e_2}a(e_1)(f)e_3 - a([e_1, e_2])(f)e_3, e_4\rangle\\
&\quad + \langle D_{e_1}fD_{e_2}(e_3) - D_{e_2}fD_{e_1}(e_3) - fD_{[e_1,e_2]}(e_3), e_4\rangle\\
&= \langle a(e_1)a(e_2)(f)e_3 - a(e_2)a(e_1)(f)e_3 - a([e_1, e_2])(f)e_3, e_4\rangle\\
&\quad + \langle a(e_2)(f)D_{e_1}e_3 - a(e_1)(f)D_{e_2}e_3 + a(e_1)(f)D_{e_2}e_3 - a(e_2)(f)D_{e_1}e_3, e_4\rangle\\
&\quad + \langle fD_{e_1}D_{e_2}(e_3) - fD_{e_2}D_{e_1}(e_3) - fD_{[e_1,e_2]}(e_3), e_4\rangle\\
&= f\Omega_{e_1,e_2}(e_3, e_4).
\end{aligned}
$$

To prove that $\Omega_{e_1,e_2}(e_3, e_4)$ is skew-symmetric on e_3, e_4 we first notice that (4.3) implies

$$\langle D_{e_1}D_{e_2}e_3, e_4\rangle - \langle e_3, D_{e_2}D_{e_1}e_4\rangle = a(e_1)(\langle D_{e_2}e_3, e_4\rangle) - a(e_2)(\langle D_{e_1}e_4, e_3\rangle) \tag{4.5}$$

and then we compute, using (4.5) and (4.3),

$$
\begin{aligned}
\Omega_{e_1,e_2}(e_3, e_4) + \Omega_{e_1,e_2}(e_4, e_3) &= \big(\langle D_{e_1}D_{e_2}e_3, e_4\rangle - \langle e_3, D_{e_2}D_{e_1}e_4\rangle\big) - \big(\langle D_{e_2}D_{e_1}e_3, e_4\rangle - \langle e_3, D_{e_1}D_{e_2}e_4\rangle\big)\\
&\quad - \big(\langle D_{[e_1,e_2]}e_3, e_4\rangle + \langle e_3, D_{[e_1,e_2]}e_4\rangle\big)\\
&= a(e_1)(\langle D_{e_2}e_3, e_4\rangle) - a(e_2)(\langle D_{e_1}e_4, e_3\rangle) - a(e_2)(\langle D_{e_1}e_3, e_4\rangle) + a(e_1)(\langle D_{e_2}e_4, e_3\rangle)\\
&\quad - a([e_1, e_2])\langle e_3, e_4\rangle\\
&= a(e_1)a(e_2)\langle e_3, e_4\rangle - a(e_2)a(e_1)\langle e_4, e_3\rangle - a([e_1, e_2])\langle e_3, e_4\rangle = 0.
\end{aligned}
$$

Properties (4.4), as well as the formula for the divergence, follow easily from (4.3) and Definition 4.1. □

***Remark* 4.28.** If E is a Courant algebroid and ∇ is an ordinary metric connection on E with curvature $F^\nabla \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 E^*)$, then $D_e := \nabla_{a(e)}$ is a generalized connection with curvature

$$\Omega_{e_1,e_2}(e_3,e_4) = F^\nabla(a(e_1),a(e_2),e_3,e_4),$$

which follows directly by definition and the fact that $a([e_1,e_2]) = [a(e_1),a(e_2)]$. In particular, $\Omega \in \Gamma(\Lambda^2 E^* \otimes \Lambda^2 E^*)$ in this case.

Example* 4.29 *(Generalized Connections on Exact Courant Algebroids). Consider $E = TM \oplus T^*M$ as a Courant algebroid with the twisted Dorfman bracket from Example 4.7:

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \iota_Y d\alpha + \iota_Y\iota_X H$$

for $H \in \Omega^3(M)$ with $dH = 0$. Let $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ be a connection on TM; it induces a connection $\nabla^* : \Gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$ as

$$X(\alpha(Y)) = \alpha(\nabla_X Y) + (\nabla^*_X\alpha)(Y).$$

By construction, $\nabla \oplus \nabla^*$ is a metric connection on $TM \oplus T^*M$ and so $D_e := (\nabla \oplus \nabla^*)_{a(e)}$ is a generalized connection. Let T_∇ be the torsion of ∇, then

$$T_D(e_1,e_2,e_3) = -(a^*H)(e_1,e_2,e_3) + \underset{1,2,3}{\mathrm{cycl}}\,\langle s(a^*T_\nabla(e_1,e_2)),e_3\rangle,$$

where $s : TM \to TM \oplus T^*M$ is $s(X) = X$. Let us show the computation: write $e_i = X_i + \alpha_i$, then

$$\begin{aligned}
T(e_1,e_2,e_3) &= \langle \nabla_{X_1}X_2 - \nabla_{X_2}X_1 - [X_1,X_2], \alpha_3\rangle + \langle \nabla_{X_3}X_1, \alpha_2\rangle \\
&\quad + \langle \nabla^*_{X_1}\alpha_2 - \nabla^*_{X_2}\alpha_1 - \mathcal{L}_{X_1}\alpha_2 + \iota_{X_2}d\alpha_1 - \iota_{X_2}\iota_{X_1}H, X_3\rangle + \langle \nabla^*_{X_3}\alpha_1, X_2\rangle \\
&= \langle T(X_1,X_2),\alpha_3\rangle + \alpha_2(\nabla_{X_3}X_1) + X_1(\alpha_2(X_3)) - \alpha_2(\nabla_{X_1}X_3) - X_2(\alpha_1(X_3)) + \alpha_1(\nabla_{X_2}X_3) \\
&\quad - d\alpha_2(X_1,X_3) - X_3(\alpha_2(X_1)) + d\alpha_1(X_2,X_3) - H(X_1,X_2,X_3) + X_3(\alpha_1(X_1)) - \alpha_1(\nabla_{X_3}X_2) \\
&= \langle T(X_1,X_2),\alpha_3\rangle - \langle T(X_1,X_3),\alpha_2\rangle + \langle T(X_2,X_3),\alpha_1\rangle - H(X_1,X_2,X_3).
\end{aligned}$$

Let $F^\nabla \in \Gamma(\Lambda^2 T^*M \otimes T^*M \otimes TM)$ be the curvature of ∇; that is, $F^\nabla(X_1,X_2) = \nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]}$. Then the curvature of D is

$$\Omega_{e_1,e_2}(e_3,e_4) = \langle s(a^*F^\nabla(e_1,e_2)e_3),e_4\rangle - \langle s(a^*F^\nabla(e_1,e_2)e_4),e_3\rangle$$

because for $e_i = X_i + \alpha_i$, $i = 1,\dots,4$ we see clearly that

$$\Omega_{e_1,e_2}(e_3,e_4) = \langle F^\nabla(X_1,X_2)X_3,\alpha_4\rangle + \langle F^{\nabla^*}(X_1,X_2)\alpha_3, X_4\rangle$$

and so the above equation is obtained by noticing that

$$\langle \nabla^*_{X_1}\nabla^*_{X_2}\alpha_3, X_4\rangle = X_1(X_2(\alpha_3(X_4))) - X_1(\alpha_3(\nabla_{X_2}X_4)) - X_2(\alpha_3(\nabla_{X_1}X_4)) + \alpha_3(\nabla_{X_2}\nabla_{X_1}X_4).$$

The divergence is simply

$$div_D(X+\alpha) = Tr(\nabla X).$$

When ∇ is torsion-free, $Tr(\nabla X) = Tr(\nabla_X - \mathcal{L}_X)$ and so for any density $\mu \in \Gamma(|det\, T^*|)$ that is parallel with respect to ∇ we have

$$div_D(X+\alpha)\mu = \mathcal{L}_X\mu,$$

which is the usual definition of the divergence of X with respect to μ.

Definition 4.30. *A **generalized metric** on a Courant algebroid E is an orthogonal decomposition $E = V_+ \oplus V_-$ such that the restriction of $\langle \cdot, \cdot \rangle$ to V_+ is non-degenerate. A generalized connection D on E is **compatible** with a generalized metric $E = V_+ \oplus V_-$ if $D(\Gamma(V_\pm)) \subset \Gamma(E^* \otimes V_\pm)$. Its torsion T is of **pure type** if $T \in \Lambda^3(\Gamma(V_+)^*) \oplus \Lambda^3(\Gamma(V_-)^*)$.*

Notice that generalized metrics do not depend on the anchor or the Dorfman bracket, they can be defined in any pseudo-Euclidean vector bundle. In the presence of a generalized connection, for $e \in \Gamma(E)$ we will write e^+ and e^- for the orthogonal projections of e onto V^+ and V^-, respectively. Notice that Proposition 4.27 implies that, for a compatible generalized connection,

$$\Omega_{e_1^+, e_2^-} = -\Omega_{e_1^-, e_2^+}, \quad \Omega_{f e_1^+, e_2^-} = f \Omega_{e_1^+, e_2^-}.$$

Definition 4.31. *The **Riemannian curvature tensors** of a generalized connection D compatible with a generalized metric $D = V_+ \oplus V_-$ are $R^\pm \in \Gamma(V_\pm^* \otimes V_\mp^* \otimes \Lambda^2 V_\pm^*)$ defined by*

$$R^+(e_1^+, e_2^-, e_3^+, e_4^+) := \Omega_{e_1^+, e_2^-}(e_3^+, e_4^+), \qquad R^-(e_1^-, e_2^+, e_3^-, e_4^-) := \Omega_{e_1^-, e_2^+}(e_3^-, e_4^-).$$

*The **Ricci tensors** $Ric^\pm \in \Gamma(V_\mp^* \otimes V_\pm^*)$ are defined by*

$$Ric^+(e_2^-, e_3^+) := Tr\left(e^+ \mapsto \Omega_{e^+, e_2^-}(e_3^+, \cdot)\right), \qquad Ric^-(e_2^+, e_3^-) := Tr\left(e^- \mapsto \Omega_{e_-, e_2^-}(e_3^+, \cdot)\right),$$

where, as usual, we are identifying $\Omega_{e^\pm, e_2^\mp}(e_3^\pm, \cdot)$ with an element in $\Gamma(V_\pm)$ through the pairing on E.

***Lemma* 4.32.** Let $E = V_+ \oplus V_-$ a generalized metric on a Courant algebroid with torsion $T \in \Gamma(\Lambda^3 E^*)$ and let D be a compatible generalized connection. Then

- If D' is any other generalized connection compatible with V^+ and with the same torsion, then $D_{e_1^\pm} e_2^\mp = D'_{e_1^\pm} e_2^\mp$ for any $e_1, e_2 \in \Gamma(E)$.
- T is of pure type if and only if $D_{e_1^-} e_2^+ = [e_1^-, e_2^+]$.

Proof.
Both statements follow directly from noticing that

$$T(e_1^-, e_2^+, e_3^+) = \langle D_{e_1^-} e_2^+ - [e_1^-, e_2^+], e_3^+ \rangle.$$

□

***Proposition* 4.33.** Given a generalized metric $E = V_+ \oplus V_-$, there exists a torsion-free generalized connection D compatible with V_+.

Proof.
Choose ordinary metric connections ∇^+ and ∇^- on V_+, V_- and set

$$D_{e_1^\pm} e_2 \pm := \nabla^\pm_{a(e_1^\pm)} e_2^\pm$$

and define $D_{e_1^{\mp}} e_2^{\pm}$ so that the torsion of D is of pure type using Lemma 4.32. Then D is clearly compatible (and it is easy to see that it is indeed a generalized connection). If T is the torsion of D, we define a new connection D^0 by killing T as

$$\langle D^0_{e_1} e_2, e_3\rangle := \langle D_{e_1} e_2, e_3\rangle - \frac{1}{3} T(e_1, e_2, e_3).$$

Because T is of pure type, D^0 is still a compatible connection and it is easy to check that it is torsion-free. □

***Remark* 4.34.** Unlike in pseudo-Riemannian geometry, there are many torsion-free generalized connections compatible with a generalized metric. Namely, if D is one such connection, the any $\chi \in \Gamma(T^3 V_+^* \oplus T^3 V_-^*)$ such that $\chi(e_1, e_2, e_3) = -\chi(e_1, e_3, e_2)$ and

$$\operatorname*{cycl}_{1,2,3} \chi(e_1, e_2, e_3) = 0$$

determines another torsion-free compatible connection D^χ as

$$\langle D^\chi_{e_1} e_2, e_3\rangle := \langle D_{e_1} e_2, e_3\rangle + \chi(e_1, e_2, e_3).$$

Example* 4.35 *(Generalized Metrics on Exact Courant Algebroids). Let V_+ be a generalized metric on the exact Courant algebroid E such that the restriction of the pairing to V_+ has positive-definite signature. Then for $0 \neq e \in \Gamma(V_+)$ we have $\langle e, e\rangle \neq 0$, so $V_+ \cap T^*M = \{0\}$ and thus by exactness the restricted anchor $a_+ : V_+ \to TM$ is an isomorphisms; let $s_0^+ = a_+^{-1} : TM \to V_+$. Then $\rho_+(X, Y) := \langle s_0^+ X, s_0^+ Y\rangle$ is a non-degenerate pairing on TM and, as in Example 4.7, the splitting $s_0^+ : TM \to V_+ \subset E$ induces an isotropic splitting $s^+(X) = s_0^+(X) - \frac{1}{2} a^* \rho_+(X, \cdot)$ and an isometry

$$\begin{aligned} \varphi_+ : TM \oplus T^*M &\to E \\ X + \alpha &\mapsto s^+(X) + a^*\alpha. \end{aligned}$$

Thus $V_+ \oplus V_-$ induces canonically the data (g, H), where $g = \rho_+$ is a Riemannian metric on M and $H \in \Gamma(\Lambda^3 T^*M)$, $H(X, Y, Z) = \langle [s^+(X), s^+(Y)], s^+(Z)\rangle$ is a preferred representative of the Ševera class of E. Note that $\varphi_+^{-1}(V_\pm) = \{X \pm g(X, \cdot) : X \in TM\}$. It is not true that any pair (g, H) determines a generalized metric because g and H must be compatible in an appropriate way. What is true is that, if we fix an isotropic splitting $s : TM \to E$, then any pair (g, b) with $b \in \Gamma(\Lambda^2 T^*M)$ does determine a generalized metric of signature $(n, 0)$ on V_+ as $V_\pm = \{s(X) + a^*(b \pm g)(X, \cdot) : X \in TM\}$, which is an easy computation.

For (g, H) corresponding to a generalized metric on E, we use the isomorphism φ_+ to identify $E \cong TM \oplus T^*M$. Let ∇^g be the Levi-Civita connection of g and define $\nabla^\pm = \nabla^g \pm \frac{1}{2} g^{-1} H$; these are connections on TM compatible with g but with non-trivial torsion. Then we claim that, for $X, Y \in \Gamma(TM)$, $[X^\mp, Y^\pm]^\pm = \pm 2(\nabla^\pm_X Y)^\pm$. Indeed,

$$\begin{aligned} &\langle [X - g(X), Y + g(Y)], Z + g(Z)\rangle = \langle [X, Y] + \mathcal{L}_X g(Y) + \iota_Y d g(X) + \iota_Y \iota_X H, Z + g(Z)\rangle \\ &\quad = g([X, Y], Z) + X(g(Y, Z)) - g([X, Z], Y) + Y(g(X, Z)) - Z(g(X, Y)) - g([Y, Z], X) + H(X, Y, Z) \\ &\quad = g(\nabla^g_X Y + \nabla^g_Y X + [X, Y], Z) + g(\nabla^g_X Z - \nabla^g_Z X - [X, Z], Y) + g(\nabla^g_Y Z - \nabla^g_Z Y - [Y, Z], X) + H(X, Y, Z) \\ &\quad = 2g(\nabla^g_X Y, Z) + H(X, Y, Z) = 2\langle \nabla^g_X Y + \frac{1}{2} g^{-1} H, Z + g(Z)\rangle \end{aligned}$$

which shows $[X^-, Y^+]^+ = 2(\nabla^+_X Y)^+$, and the other one is similar. It follows from Lemma 4.32 that any connection D^B on E with torsion of pure type must satisfy

$$D^B_{X^-} Y^+ = 2(\nabla^+_X Y)^+, \qquad D^B_{X^+} Y^- = -2(\nabla^-_X Y)^-$$

A natural way to complete the definition of D^B is then

$$D^B_{X^+} Y^+ = 2(\nabla^+_X Y)^+, \qquad D^B_{X^-} Y^- = -2(\nabla^-_X Y)^-;$$

that is, $D^B_e = 2(\nabla^+ \oplus -\nabla^-)_{a(e)}$, which is known in the literature as the **Gaultieri-Bismut** connection [19]. It has torsion $T_{D^B} = a^*_+ H - a^*_- H$, for $a_\pm : V_\pm \to TM$ the isomorphisms obtained by restricting the anchor, and its curvature can be identified with the curvatures of $\nabla^\pm$ as in Remark 4.28. As in the proof of Proposition 4.33, $D^0 := D^B - \frac{1}{3} T_{D^B}$ is a torsion-free generalized connection compatible with V_+. Its pure-type operators are

$$D^0_{X^+} Y^+ = 2(\nabla^{+1/3}_X Y)^+, \qquad D^0_{X^-} Y^- = 2(\nabla^{-1/3}_X Y)^-,$$

for $\nabla^{+1/3} := \nabla^g \pm \frac{1}{6} g^{-1} H$ and its Riemannian curvature tensor R^+ is [18]

$$\begin{aligned} R^+(X^+, Y^-, Z^+, \cdot) = \frac{1}{4} R^g(X,Y)Z + g^{-1}\Bigg(& \frac{1}{2}(\nabla^g_X H)(Y,Z,\cdot) - \frac{1}{6}(\nabla^g_Y H)(X,Z,\cdot) \\ & + \frac{1}{12} H(X, g^{-1} H(Y,Z,\cdot),\cdot) - \frac{1}{12} H(Y, g^{-1} H(X,Z,\cdot),\cdot) - \frac{1}{6} H(Z, g^{-1} H(X,Y,\cdot),\cdot)\Bigg), \end{aligned}$$

where R^g is the curvature of ∇^g. What is interesting about this is that, as we will see in Corollary 4.38 below, this complicated expression satisfies a very simple first Bianchi identity.

4.6. Generalized Riemannian Geometry & Graded Geometry

In this section we construct a graded Poisson manifold from a Courant algebroid E with a generalized connection D and we interpret some of the objects from generalized Riemannian geometry presented in Section 4.5 by means of this graded manifold. In particular, we interpret the master equation as a **first Bianchi identity** for the curvature of a generalized connection and we construct a **Morita equivalence** of graded Poisson manifolds from a generalized metric.

Consider a generalized connection D on a Courant algebroid E and write $\mathbb{A} = \{D_e : e \in \Gamma(E)\}$ for the vector bundle of covariant derivatives with respect to D. We define the graded manifold

$$\mathcal{M}^D := (M, \Gamma(\Lambda^* E \otimes S^* \mathbb{A})/I),$$

where elements of $\Gamma(E)$ have degree 1, elements of $\mathbb{A}$ have degree 2 and I is the ideal generated by $\{1 \otimes D_e - D_e \otimes 1 \;:\; e \in Ker\, a\}$, taking into account that for e such that $a(e) = 0$ condition (4.3) means that we can identify D_e with a skew-symmetric operator on E acting as

$$D_e(s_1, s_2) := -\langle D_e s_1, s_2 \rangle.$$

The reason why we write this minus sign will become apparent later. This is indeed a well-defined graded manifold at least when the anchor has constant rank because we can think of it as $E[1] \oplus (E/Ker\, a)[2]$, where we identify each $\bar{e} \in \Gamma(E/Ker\, a)$ with the covariant derivative D_e. The torsion T of D is an element of $\Gamma(\Lambda^3 E^*)$; hence, a function of degree 3 on $\mathcal{M}^D$. For fixed $e_1, e_2 \in \Gamma(E)$, the curvature $\Omega_{e_1,e_2} \in \Gamma(\Lambda^2 E^*)$ is a function of degree 2. Finally, the connection itself is an element of $\Gamma(E^* \otimes \mathbb{A})$; hence, it defines a function $\bar{D} \in C^\infty_3(\mathcal{M}^D)$.

Theorem 4.36. *The following relations can be extended using Leibniz's rule to a degree -2 Poisson bracket on $\mathcal{M}^D$: For $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,*

$$\{e_1, e_2\} = \langle e_1, e_2\rangle, \qquad \{D_{e_1}, f\} = a(e_1)(f),$$
$$\{D_{e_1}, e_2\} = D_{e_1}e_2, \qquad \{D_{e_1}, D_{e_2}\} = D_{[e_1,e_2]} - \Omega_{e_1,e_2}.$$

This bracket is non-degenerate if and only if the Courant algebroid E is transitive. Moreover, for $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\{\{\bar{D}+T, e_1\}, f\} = a(e_1)(f), \quad \{\{\bar{D}+T, e_1\}, e_2\} = [e_1, e_2], \quad \{\bar{D}+T, f\} = d_E f \tag{4.6}$$

and

$$\{\bar{D}+T, \bar{D}+T\} = 0.$$

Proof.

This proof is analogous to those of Proposition 4.13 and Theorem 4.20, just paying attention to some peculiarities of generalized connections. Let us check first that Leibniz's rule is satisfied:

$$\{D_{e_1}, fg\} = a(e_1)(fg) = a(e_1)(f)g + f a(e_1)(g) = \{D_{e_1}, f\}g + f\{D_{e_1}, g\},$$
$$\{D_{e_1}, fe_2\} = D_{e_1}(fe_2) = a(e_1)fe_2 + fD_{e_1}e_2 = \{D_{e_1}, f\}e_2 + f\{D_{e_1}, e_2\},$$
$$\{D_{e_1}, fD_{e_2}\} = D_{[e_1,fe_2]} - \Omega_{e_1,fe_2} = fD_{[e_1,e_2]} + a(e_1)(f)D_{e_2} - f\Omega_{e_1,e_2} = f\{D_{e_1}, D_{e_2}\} + \{D_{e_1}, f\}D_{e_2},$$
$$\{fD_e, g\} = a(fe)(g) = fa(e)(g) = f\{D_e, g\} + \{f, g\}D_e,$$
$$\{fD_{e_1}, e_2\} = D_{fe_1}e_2 = fD_{e_1}e_2 = f\{D_{e_1}, e_2\} + \{f, e_2\}D_{e_1}.$$

These relations extend through Leibniz's rule as $\{e_1, \alpha\} = \iota_{e_1}\alpha$ for $e_1 \in \Gamma(E)$, $\alpha \in \Gamma(\Lambda^* E^*)$ simply because ι_{e_1} and $\{e_1, \cdot\}$ are both derivations of degree -1 of the algebra $\Gamma(\Lambda^* E)$ coinciding on $\Gamma(E)$, which generates the whole algebra, and similarly $\{D_{e_1}, \alpha\} = D_{e_1}\alpha$. In particular, for $\alpha \in \Gamma(\Lambda^2 E^*)$, we have $\{\{\alpha, e_1\}, e_2\} = -\alpha(e_1, e_2)$, while $\{\{D_e, s_1\}, s_2\} = \langle D_e s_1, s_2\rangle$. This is the reason why we used a minus sign to define the action of $D_e \in \Gamma(\Lambda^2 E)$ when $a(e) = 0$, so that this bracket is well-defined on elements of the ideal I.

It is clear from the definition that, for a fixed $f \in C^\infty(M)$, $\{H, f\} = 0, \forall H \in C^\infty(\mathcal{M}^D)$ if and only if f is constant along the image of the anchor. Hence, this bracket is degenerate for non-transitive Courant algebroids. However, for fixed $F \in C^\infty_{\geq 1}(\mathcal{M}^D)$ it is true that $\{H, F\} = 0, \forall H \in C^\infty(\mathcal{M}^D)$ implies $F = 0$. This is clear in degree 1 and, for $F \in C^\infty_2(\mathcal{M}^D)$, $F = D_e + \alpha$ for some $e \in \Gamma(E)$ and $\alpha \in \Gamma(\Lambda^2 E^*)$, so $\{F, f\} = a(e)(f) = 0$ $\forall f \in C^\infty(M)$ implies that we can identify F with an element of $\Gamma(\Lambda^2 E)$ and then $\{F, e\} = -\iota_e F = 0$ $\forall e \in \Gamma(E)$ would imply that $F = 0$. This shows that the Poisson bracket is non-degenerate for functions of degree 1 and 2 and it leads to its non-degeneracy in higher degrees because these functions generate all of $C^\infty(\mathcal{M}^D)$.

Now $\{e_1, e_2\} = \{e_2, e_1\}$ is clear and $\{D_{e_1}, D_{e_2}\} = -\{D_{e_2}, D_{e_1}\}$ follows from Proposition 4.27, so we proceed to prove the Jacobi identity for this bracket. First,

$$\{D_{e_1}, \{e_2, e_3\}\} = a(e_1)\langle e_2, e_3\rangle = \langle D_{e_1}e_2, e_3\rangle + \langle e_2, D_{e_1}e_3\rangle = \{\{D_{e_1}, e_2\}, e_3\} + \{e_2, \{D_{e_1}, e_3\}\},$$
$$\{\{D_{e_1}, D_{e_2}\}, f\} = a([e_1, e_2])(f) = [a(e_1), a(e_2)](f) = \{\{D_{e_1}, f\}, D_{e_2}\} + \{D_{e_1}\{D_{e_2}, f\}\},$$
$$\{\{D_{e_1}, D_{e_2}\}, e_3\} = D_{[e_1,e_2]}e_3 + \Omega_{e_1,e_2}(\cdot, e_3) = D_{e_1}D_{e_2}e_3 - D_{e_2}D_{e_1}e_3 = \{\{D_{e_1}, e_3\}, D_{e_2}\} + \{D_{e_1}, \{D_{e_2}, e_3\}\}$$

It only remains to prove the Jacobi identity for $\{\{D_{e_1}, D_{e_2}\}, D_{e_3}\}$. Notice that $\{D_{e_1}, D_{e_2}\}$ is, by definition, the unique function $H \in C_2^\infty(\mathcal{M}^D)$ such that $\{H, f\} = a([e_1, e_2])(f)$ and $\{\{H, s_1\}, s_2\} = \langle [D_{e_1}, D_{e_2}]s_1, s_2\rangle$, where $[D_{e_1}, D_{e_2}]$ denotes the commutator of the operators D_{e_1}, D_{e_2} and $s_1, s_2 \in \Gamma(E)$. Thus,

$$\{\{D_{e_1}, D_{e_2}\}, D_{e_3}\} = \{D_{[e_1,e_2]}, D_{e_3}\} - \{\Omega_{e_1,e_2}, D_{e_3}\}$$

is the unique $H \in \mathcal{A}^2$ such that

$$\{H, f\} = a([[e_1, e_2], e_3])(f) \quad \text{and} \quad \{\{H, s_1\}, s_2\} = \langle [D_{[e_1,e_2]}, D_{e_3}]s_1, s_2\rangle - \{\{\{\Omega_{e_1,e_2}, D_{e_3}\}, s_1\}, s_2\}.$$

We claim that the last term is $\langle [[D_{e_1}, D_{e_2}], D_{e_3}]s_1, s_2\rangle$ (this is essentially a second Bianchi identity for D), which will conclude the proof because both the Dorfman bracket and the commutator satisfy the Jacobi identity. Indeed,

$$[D_{[e_1,e_2]}, D_{e_3}] = [[D_{e_1}, D_{e_2}], D_{e_3}] + [\Omega_{e_1,e_2}, D_{e_3}],$$

where we are simply interpreting Ω_{e_1,e_2} as an operator sending s_1 to $\Omega_{e_1,e_2}(\cdot, s_1)$ and, as such, we see that

$$\begin{aligned}\langle [\Omega_{e_1,e_2}, D_{e_3}]s_1, s_2\rangle &= \Omega_{e_1,e_2}(s_2, D_{e_3}s_1) - \langle D_{e_3}(\Omega_{e_1,e_2}(\cdot, s_1)), s_2\rangle \\ &= \Omega_{e_1,e_2}(s_2, D_{e_3}s_1) - a(e_3)(\Omega_{e_1,e_2}(s_2, s_1)) + \Omega_{e_1,e_2}(D_{e_3}s_2, s_1) \\ &= (D_{e_3}\Omega_{e_1,e_2})(s_1, s_2) = \{\{\{\Omega_{e_1,e_2}, D_{e_3}\}, s_1\}, s_2\},\end{aligned}$$

which proves the claim. In order to obtain the identities (4.6) we choose a local frame $\{\xi^i\}_i$ of E with dual frame $\{\tilde{\xi}^i\}_i$ (that is, $\langle \xi^i, \tilde{\xi}^j\rangle = \delta_{i,j}$) so that we can write $\bar{D} = \tilde{\xi}^i D_{\xi^i}$. Then

$$\{\{\bar{D}, e_1\}, f\} = \{\tilde{\xi}^i(e_1)D_{\xi^i}, f\} + \{\tilde{\xi}^i D_{\xi^i}e_1, f\} = \{D_{e_1}, f\} = a(e_1)(f),$$
$$\{\{\bar{D}, e_1\}, e_2\} = \{\tilde{\xi}^i(e_1)D_{\xi^i}, e_2\} + \{\tilde{\xi}^i D_{\xi^i}e_1, e_2\} = D_{e_1}e_2 + \tilde{\xi}^i\langle D_{\xi^i}e_1, e_2\rangle - D_{e_2}e_1 = T(e_1, e_2, \cdot) + [e_1, e_2];$$

the second equation and the fact that $\{\{T, e_1\}, e_2\} = T(\cdot, e_2, e_1) = -T(e_1, e_2, \cdot)$ imply $\{\{\bar{D} + T, e_1\}, e_2\} = [e_1, e_2]$. Thus $\{\bar{D} + T, f\} \in \Gamma(E)$ is such that

$$\{\{\bar{D} + T, f\}, e_1\} = \{\{\bar{D} + T, e_1\}, f\} = a(e_1)(f),$$

which is precisely what $\{\bar{D} + T, f\} = d_E f$ means. Finally, these relations imply that $\{\bar{D} + T, \bar{D} + T\} = 0$ in the same way as in Theorem 4.20; we just need to check that any $H \in C_4^\infty(\mathcal{M}^D)$ satisfying

$$\{\{H, f\}, g\} = 0, \qquad \{\{\{H, e_1\}, e_2\}, f\} = 0, \qquad \{\{\{\{H, e_1\}, e_2\}, e_3\}, e_4\} = 0$$

for all $f, g \in C^\infty(M)$ and $e_1, e_2, e_3, e_4 \in \Gamma(E)$ must be $H = 0$. This can be seen by writing $H = D_{e_1} D_{e_2} + D_{e_3}\alpha + \beta$ with $\alpha \in \Gamma(\Lambda^2 E^*)$ and $\beta \in \Gamma(\Lambda^4 E^*)$. Then $\{\{H, f\}, g\} = 0\, \forall f, g \in C^\infty(M)$ implies precisely that $D_{e_1} D_{e_2} \in I$ and so we can identify H with something in $\Gamma(\Lambda^2 E \otimes \mathbb{A} \oplus \Lambda^4 E)$, but then $\{\{\{H, e_1\}, e_2\}, f\} = 0$ implies that H can be identified with something in $\Gamma(\Lambda^4 E)$ and finally $\{\{\{\{H, e_1\}, e_2\}, e_3\}, e_4\} = 0$ means $H = 0$. □

The relation between $\mathcal{M}^D$ and the canonical symplectic graded manifold $\mathcal{M}$ constructed in Section 4.3 is the following. For a generalized connection D and an ordinary metric connection ∇ related by $D_e = \nabla_{a(e)}$, we can define a map $\mathcal{M}^\nabla \cong E[1] \oplus T[2]M \to \mathcal{M}^D \cong E[1] \oplus (E/Ker\, a)[2]$ with pull-back

$$\begin{aligned} C^\infty(\mathcal{M}^D) &\to C^\infty(\mathcal{M}^\nabla) \\ f &\mapsto f, \\ e_1 &\mapsto e_1, \\ D_{e_1} &\mapsto a(e_1). \end{aligned}$$

It is clear from the way the Poisson bracket is defined on each graded manifold that this is a Poisson map. In fact, $E/Ker\, a \cong Im\, a$, and so for transitive Courant algebroids (those with $Im\, a = TM$) this is a symplectomorphism of graded manifolds. However, for non-transitive Courant algebroids this is a surjective submersion, showing that $\mathcal{M}^\nabla$ is a **symplectic realization** of $\mathcal{M}^D$.

The advantage of working with $\mathcal{M}^D$ is that some of the objects from generalized Riemannian geometry have a nice interpretation here. For example, recall that the curvature $\Omega_{e_1,e_2}(e_3, e_4)$ of a generalized connection D on a Courant algebroid E is not a skew-symmetric tensor on e_1, e_2. Instead, it satisfies the relations in Proposition 4.27. We can interpret these in $\mathcal{M}^D$ as follows: For each fixed e_3, e_4, there exists a unique function $\Omega(e_3, e_4) \in C^\infty_2(\mathcal{M}^D)$ such that

$$\begin{aligned} \{\{\Omega(e_3, e_4), e_1\}, e_2\} &= -\Omega_{e_1,e_2}(e_3, e_4), \\ \{\Omega(e_3, e_4), f\} &= \langle D_{d_E f} e_3, e_4 \rangle. \end{aligned}$$

In a local frame $\{\xi^i\}_i$ with dual frame $\{\tilde{\xi}^i\}_i$ we can write this function as

$$\Omega(e_3, e_4) = \frac{1}{2}\tilde{\xi}^i \tilde{\xi}^j \left(\Omega_{\xi^i,\xi^j}(e_3, e_4) - \langle D_{\xi^k} e_3, e_4\rangle\langle D_{\tilde{\xi}^k}\xi^i, \xi^j\rangle\right) + \langle D_{\xi^i} e_3, e_4\rangle D_{\tilde{\xi}^i}.$$

Thus, we may see the curvature of D as a $C^\infty(M)$-linear map $\Omega : \Gamma(\Lambda^2 E) \to C^\infty_2(\mathcal{M}^D)$, $(e_3, e_4) \mapsto \Omega(e_3, e_4)$. In other words, in general Ω is an element of $C^\infty_2(\mathcal{M}^D) \otimes \Gamma(\Lambda^2 E^*)$ instead of $\Gamma(\Lambda^2 E^*) \otimes \Gamma(\Lambda^2 E^*)$, which is what happens when $D_e = \nabla_{a(e)}$ for a metric connection ∇ (see Remark 4.28).

Theorem 4.37 (First Bianchi Identity for Generalized Connections). *The following identity is satisfied by the curvature and torsion of a generalized connection:*

$$\begin{aligned} \underset{1,2,3,4}{cycl}\ \Omega_{e_1,e_2}(e_3, e_4) = -2 \sum_{i_1<i_2, i_3<i_4} (-1)^\gamma \langle D_{\xi^i} e_{i_1}, e_{i_2}\rangle\langle D_{\tilde{\xi}^i} e_{i_3}, e_{i_4}\rangle \\ + \frac{2}{3} \underset{1,2,3,4}{cycl}\ D_{e_1} T(e_2, e_3, e_4) + \frac{1}{2} \underset{1,2,3,4}{cycl}\ T(e_1, e_2, T(e_3, e_4, \cdot)). \end{aligned}$$

Proof.

This identity can be obtained by expanding $\{\{\{\{\{\bar{D}+T, \bar{D}+T\}, e_1\}, e_2\}, e_3\}, e_4\} = 0$. In a local frame $\{\xi^i\}_i$ with dual frame $\{\tilde{\xi}^i\}_i$, we see

$$\{\bar{D}, \bar{D}\} = D_{\xi^i} \cdot D_{\tilde{\xi}^i} + 2\tilde{\xi}^j \cdot D_{\xi^j}\xi^i \cdot D_{\tilde{\xi}^i} + \tilde{\xi}^i \cdot \tilde{\xi}^j \cdot (D_{[\xi^i,\xi^j]} - \Omega_{\xi^i,\xi^j}).$$

Now

$$\{\{\{\{D_{\xi^i} \cdot D_{\tilde{\xi}^i}, e_1\}, e_2\}, e_3\}, e_4\} = \sum_{i_1<i_2, i_3<i_4} (-1)^\gamma \langle D_{\xi^i} e_{i_1}, e_{i_2}\rangle\langle D_{\tilde{\xi}^i} e_{i_3}, e_{i_4}\rangle,$$
$$\{\{\{\{2\tilde{\xi}^j \cdot D_{\xi^j}\xi^i \cdot D_{\tilde{\xi}^i}, e_1\}, e_2\}, e_3\}, e_4\} = -2\sum_{i_3<i_4} (-1)^\gamma \langle D_{e_{i_1}}\xi^i, e_{i_2}\rangle\langle D_{\tilde{\xi}^i} e_{i_3}, e_{i_4}\rangle.$$

We note that

$$\begin{aligned}
&(\tilde{\xi}^i(e_1)\tilde{\xi}^j(e_2) - \tilde{\xi}^i(e_2)\tilde{\xi}^j(e_1))[\xi^i, \xi^j] \\
&\quad = [e_1, e_2] - [e_2, e_1] + \tilde{\xi}^i(e_1)d_E(\xi^i(e_2)) - \tilde{\xi}^i(e_2)d_E(\xi^i(e_1)) + 2a(e_2)(\tilde{\xi}^i(e_1))\xi^i - 2a(e_1)(\tilde{\xi}^i(e_2))\xi^i \\
&\quad = 2[e_1, e_2] - 2\tilde{\xi}^i(e_2)d_E(\xi^i(e_1)) + 2\langle D_{e_2}\tilde{\xi}^i, e_1\rangle\xi^i + 2D_{e_2}e_1 - 2\langle D_{e_1}\tilde{\xi}^i, e_2\rangle\xi^i - 2D_{e_1}e_2 \\
&\quad = -2T(e_1, e_2, \cdot) + 2\tilde{\xi}^i\langle D_{\xi^i}e_1, e_2\rangle - 2\tilde{\xi}^i(e_2)d_E(\xi^i(e_1)) + 2\langle D_{e_2}\tilde{\xi}^i, e_1\rangle\xi^i - 2\langle D_{e_1}\tilde{\xi}^i, e_2\rangle\xi^i,
\end{aligned}$$

while

$$\begin{aligned}
&(\tilde{\xi}^i(e_1)\tilde{\xi}^j(e_2) - \tilde{\xi}^i(e_2)\tilde{\xi}^j(e_1))\Omega_{\xi^i,\xi^j}(e_3, e_4) \\
&\quad = \Omega_{e_1,e_2}(e_3, e_4) + \langle D_{\tilde{\xi}^i(e_2)d_E(\xi^i(e_1))}e_3, e_4\rangle - \Omega_{e_2,e_1}(e_3, e_4) - \langle D_{\tilde{\xi}^i(e_1)d_E(\xi^i(e_2))}e_3, e_4\rangle \\
&\quad = 2\Omega_{e_1,e_2}(e_3, e_4) + 2\langle D_{\tilde{\xi}^i(e_2)d_E(\xi^i(e_1))}e_3, e_4\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\{\{\{\{\tilde{\xi}^i \cdot \tilde{\xi}^j \cdot (D_{[\xi^i,\xi^j]} - \Omega_{\xi^i,\xi^j}), e_1\}, e_2\}, e_3\}, e_4\} \\
&\quad = -\sum_{i_1<i_2,i_3<i_4} (-1)^\gamma \left(\tilde{\xi}^i(e_{i_1})\tilde{\xi}^j(e_{i_2}) - \tilde{\xi}^i(e_{i_2})\tilde{\xi}^j(e_{i_1})\right)\left(\langle D_{[\xi^i,\xi^j]}e_{i_3}, e_{i_4}\rangle + \Omega_{\xi^i,\xi^j}(e_{i_3}, e_{i_4})\right) \\
&\quad = 2\sum_{i_1<i_2,i_3<i_4} (-1)^\gamma \left(\langle D_{T(e_{i_1},e_{i_2},\cdot)}e_{i_3}, e_{i_4}\rangle - \langle D_{\xi^i}e_{i_1}, e_{i_2}\rangle\langle D_{\tilde{\xi}^i}e_{i_3}, e_{i_4}\rangle - \Omega_{e_{i_1},e_{i_2}}(e_{i_3}, e_{i_4})\right) \\
&\qquad + 2\sum_{i_3<i_4} (-1)^\gamma \langle D_{e_{i_1}}\xi^i, e_{i_2}\rangle\langle D_{\tilde{\xi}^i}e_{i_3}, e_{i_4}\rangle.
\end{aligned}$$

That is,

$$\begin{aligned}
&\{\{\{\{\{\bar{D}, \bar{D}\}, e_1\}, e_2\}, e_3\}, e_4\} = \\
&\qquad \sum_{i_1<i_2,i_3<i_4} (-1)^\gamma \left(2\langle D_{T(e_{i_1},e_{i_2},\cdot)}e_{i_3}, e_{i_4}\rangle - \langle D_{\xi^i}e_{i_1}, e_{i_2}\rangle\langle D_{\tilde{\xi}^i}e_{i_3}, e_{i_4}\rangle - 2\Omega_{e_{i_1},e_{i_2}}(e_{i_3}, e_{i_4})\right).
\end{aligned}$$

On the other hand,

$$\{\bar{D}, T\} = \tilde{\xi}^i \cdot D_{\xi^i}T + T(\tilde{\xi}^i, \cdot, \cdot) \cdot D_{\xi^i},$$

so

$$2\{\{\{\{\{\bar{D},T\},e_1\},e_2\},e_3\},e_4\} = 2\sum_{i_2<i_3<i_4}(-1)^\gamma(D_{e_{i_1}}T)(e_{i_2},e_{i_3},e_{i_4}) - 2\sum_{i_1<i_2,i_3<i_4}(-1)^\gamma T(\tilde{\xi}^i,e_{i_1},e_{i_2})\langle D_{\xi^i}e_{i_3},e_{i_4}\rangle$$
$$= 2\sum_{i_2<i_3<i_4}(-1)^\gamma(D_{e_{i_1}}T)(e_{i_2},e_{i_3},e_{i_4}) - 2\sum_{i_1<i_2,i_3<i_4}(-1)^\gamma\langle D_{T(e_{i_1},e_{i_2},\cdot)}e_{i_3},e_{i_4}\rangle$$

Finally, $\{T,T\} = T(\xi^i,\cdot,\cdot)\cdot T(\cdot,\cdot,\tilde{\xi}^i)$ implies

$$\{\{\{\{\{T,T\},e_1\},e_2\},e_3\},e_4\} = \sum_{i_1<i_2,i_3<i_4}(-1)^\gamma T(\xi^i,e_{i_1},e_{i_2})T(e_{i_3},e_{i_4},\tilde{\xi}^i) = \sum_{i_1<i_2,i_3<i_4}(-1)^\gamma T(e_{i_1},e_{i_2},T(e_{i_3},e_{i_4},\cdot)).$$

Thus,

$$\{\{\{\{\{\bar{D}+T,\bar{D}+T\},e_1\},e_2\},e_3\},e_4\}$$
$$= \sum_{i_1<i_2,i_3<i_4}(-1)^\gamma\Big(-2\Omega_{e_{i_1},e_{i_2}}(e_{i_3},e_{i_4}) - \langle D_{\xi^i}e_{i_1},e_{i_2}\rangle\langle D_{\tilde{\xi}^i}e_{i_3},e_{i_4}\rangle + T(e_{i_1},e_{i_2},T(e_{i_3},e_{i_4},\cdot))\Big)$$
$$+\,2\sum_{i_2<i_3<i_4}(-1)^\gamma(D_{e_{i_1}}T)(e_{i_2},e_{i_3},e_{i_4}),$$

To conclude the proof notice that

$$\Omega_{e_1,e_2}(e_3,e_4) - \Omega_{e_2,e_1}(e_3,e_4) - \Omega_{e_1,e_2}(e_4,e_3) + \Omega_{e_2,e_1}(e_4,e_3) = 4\Omega_{e_1,e_2}(e_3,e_4) + 2\langle D_{d_E\langle e_1,e_2\rangle}e_3,e_4\rangle$$

and

$$\langle D_{d_E\langle e_1,e_2\rangle}e_3,e_4\rangle - \langle D_{\langle D_{\xi^i}e_1,e_2\rangle\tilde{\xi}^i}e_3,e_4\rangle = \langle D_{\langle D_{\xi^i}e_2,e_1\rangle\tilde{\xi}^i}e_3,e_4\rangle.$$

□

The Bianchi identity in Theorem 4.37 is analogous to the Bianchi identity for an ordinary connection with non-vanishing torsion (see for example [30]) except for a term that appears as a consequence of the non skew-symmetry of Ω_{e_1,e_2} on e_1, e_2. In the presence of a compatible generalized metric, it has the following much more elegant Corollary which was proved in [19] for the case $T = 0$.

***Corollary* 4.38.** Let D be a generalized connection on a Courant algebroid E compatible with the metric $E = V_+ \oplus V_-$ and with torsion T of pure type. Then,

$$R^\pm(e_1^\pm,e^\mp,e_2^\pm,e_3^\pm) - R^\pm(e_2^\pm,e^\mp,e_1^\pm,e_3^\pm) + R^\pm(e_3^\pm,e^\mp,e_1^\pm,e_2^\pm) = -D_{e^\mp}T(e_1^\pm,e_2^\pm,e_3^\pm)$$

Proof.
This follows directly from Theorem 4.37, using the compatibility of D and the orthogonality of V_+ and V_-. □

Generalized metrics admit a geometric interpretation on $\mathcal{M}^\nabla$ and $\mathcal{M}^D$. Namely, it follows from Remark 4.15 that a generalized metric on a pseudo-Euclidean vector bundle $E \to M$ is precisely a symplectic submanifold $\mathcal{N}_+$ of its corresponding symplectic N-manifold $\mathcal{M}$ having M as underlying manifold. If the generalized metric is given as

$E = V_+ \oplus V_-$, then $\mathcal{N}_+$ is simply the symplectic N-manifold corresponding to the pseudo-Euclidean vector bundle V_+. If ∇ is a metric connection on E compatible with $E = V_+ \oplus V_-$, then we can present these manifolds as

$$\mathcal{M}^\nabla = (M, \Gamma(S^*TM \otimes \Lambda^* E)), \qquad \mathcal{N}_\pm^\nabla = (M, \Gamma(S^*TM \otimes \Lambda^* V_\pm))$$

with the Poisson brackets from Section 4.2. Interestingly, we see that coordinates on $\mathcal{M}^\nabla$ corresponding to $\Gamma(V_-)$ transform linearly between themselves (independently of the coordinates corresponding to $\Gamma(V_+)$). This means that $\mathcal{M}^\nabla \to \mathcal{N}_+^\nabla$ is a vector bundle projection with fiber the model vector space of V_-, and similarly for $\mathcal{M}^\nabla \to \mathcal{N}_-^\nabla$. The global picture is

$$\begin{tikzcd}
 & \mathcal{M}^\nabla \arrow[dl] \arrow[dr] & \\
\mathcal{N}_+^\nabla \arrow[d] & & \mathcal{N}_-^\nabla \arrow[d] \\
V_+[1] \arrow[dr] & & V_-[1] \arrow[dl] \\
 & M &
\end{tikzcd},$$

where the middle arrows are affine bundle projections, the rest are vector bundle projections and all maps are Poisson maps. Note that the fibres of the projection π_+ are symplectically orthogonal to the fibres of the projection π_-. This kind of structure is usually called in ordinary Poisson geometry a **Morita equivalence** between $\mathcal{N}_+^\nabla$ and $\mathcal{N}_-^\nabla$.

If E is a Courant algebroid and we want to work with a generalized connection D compatible with $E = V_+ \oplus V_-$, we can construct the graded manifolds $\mathcal{N}_\pm^D := (M, \Gamma(\Lambda^* V_\pm \otimes S^* \mathbb{A})/I^\pm)$, where $I^\pm$ is the ideal generated by $\{1 \otimes D_e - D_e^\pm \otimes 1 \,:\, a(e) = 0\}$ and $D_e^\pm$ is the restriction of $D_e \in \Gamma(\Lambda^2 V_+^* \oplus \Lambda^2 V_-^*)$ to $\Gamma(\Lambda^2 V_\pm^*)$ when $a(e) = 0$. That is, $\mathcal{N}_\pm^D = V_\pm[1] \oplus (E/Ker\, a)[2]$. As before, these are graded Poisson manifolds and there are Poisson vector bundle projections $\pi_\pm : \mathcal{M}^D \to \mathcal{N}_\pm^D$.

Consider the injections $i_\pm : \mathcal{N}_\pm^D \to \mathcal{M}^D$ induced from the orthogonal projections $E \to V_\pm$ (these are injections along the zero section of $\mathcal{M}^D$). We emphasize that these are not Poisson maps; for non-orthogonal $e_1^-, e_2^- \in C^\infty(\mathcal{M}^D)$ we have $\{i_+^* e_1^-, i_+^* e_2^-\} = 0 \neq i^*\{e_1^-, e_2^-\}$. For any $H \in C^\infty(\mathcal{M}^D)$ we write $H^\pm := \pi_\pm^* i_\pm^* H$. In particular, for $\alpha \in \Gamma(\Lambda^p E^*)$, $\alpha^\pm \in \Gamma(\Lambda^p V_\pm^*)$ is its restriction to $V_\pm$ and, for $\{\xi_+^i, \xi_-^j\}_{i,j}$ a local basis of $\Gamma(E) = \Gamma(V_+) \oplus \Gamma(V_-)$ with dual basis $\{\tilde{\xi}_+^i, \tilde{\xi}_-^j\}_{i,j}$, we can write

$$\bar{D}^+ = \tilde{\xi}_+^i D_{\xi_+^i}, \qquad \bar{D}^- = \tilde{\xi}_-^j D_{\xi_-^j};$$

i.e., $\bar{D}^\pm \in \Gamma(V_+^* \otimes \mathbb{A})$ is the restriction of $D \in \Gamma(E^* \otimes \mathbb{A})$ to $V_\pm$.

***Proposition* 4.39.** Let $E = V_+ \oplus V_-$ be a Courant algebroid with a generalized metric and let D be a compatible generalized connection with torsion of pure type. Then, for $\Theta \in C^\infty(\mathcal{M}^D)$ defined by $\Theta = D + T$ we have

$$\{\Theta^+, \Theta^-\} = 0.$$

Proof.
Performing similar computations to those in the proof of Theorem 4.37 one can check that

$$\begin{aligned}\{\{\{\{\{\bar{D}^+,\bar{D}^-\},e_1\},e_2\},e_3\},e_4\}\\ &= \sum_{i_1<i_2,i_3<i_4}(-1)^\gamma\left(\langle D_{T(e^+_{i_1},e^-_{i_2},\cdot)+T(e^-_{i_1},e^+_{i_2},\cdot)}e_{i_3},e_{i_4}\rangle - \Omega_{e^+_{i_1},e^-_{i_2}}(e_{i_3},e_{i_4}) - \Omega_{e^-_{i_1},e^+_{i_2}}(e_{i_3},e_{i_4})\right)\end{aligned}$$

and it is also easy to see that

$$\begin{aligned}\{\{\{\bar{D}^+,\bar{D}^-\},f\},g\} &= 0,\\ \{\{\{\{\bar{D}^+,\bar{D}^-\},e_1\},e_2\},f\} &= a(T(e^+_1,e^-_2,\cdot)+T(e^-_1,e^+_2,\cdot))(f).\end{aligned}$$

When T is of pure type, we note that

$$\{T^+,T^-\}=0,\qquad \{\bar{D}^+,T^-\}=\tilde{\xi}^i_+\cdot D_{\xi^i_+}T^-,\qquad \{T^+,\bar{D}^-\}=\tilde{\xi}^i_-\cdot D_{\xi^i_-}T^+,$$

implying

$$\begin{aligned}\{\{\{\bar{D}^++T^+,\bar{D}^-+T^-\},f\},g\} &= \{\{\{\bar{D}^+,\bar{D}^-\},f\},g\}=0,\\ \{\{\{\{\bar{D}^++T^+,\bar{D}^-+T^-\},e_1\},e_2\},f\} &= \{\{\{\{\bar{D}^+,\bar{D}^-\},e_1\},e_2\},f\}=0.\end{aligned}$$

Now using Corollary 4.38 we obtain in this case

$$\begin{aligned}\{\{\{\{\{\bar{D}^+,\bar{D}^-\},e_1\},e_2\},e_3\},e_4\} &= -\sum_{i_2<i_3<i_4}(-1)^\gamma\left(D_{e^-_{i_1}}T(e^+_{i_2},e^+_{i_3},e^+_{i_4})+D_{e^+_{i_1}}T(e^-_{i_2},e^-_{i_3},e^-_{i_4})\right)\\ &= -\{\{\{\{\{\bar{D}^+,T-\}+\{T^-,\bar{D}^+\},e_1\},e_2\},e_3\},e_4\},\end{aligned}$$

which means precisely that $\{\Theta^+,\Theta^-\}=0$ for $\Theta^\pm=\bar{D}^\pm+T^\pm$. □

The equation $\{\Theta^+,\Theta^-\}=0$ holds a strong resemblance with the equation $\{\Theta_A,\Theta_{A^*}\}=0$ from the study of Dirac structures on the double of a Lie bialgebroid in Example 4.25, but its real significance remains a mistery to us.

Finally, we mention that some work has been carried out in [2], [54] trying to study generalized connections as **Q-connections**. These can be defined for any graded vector bundle $\mathcal{E}\to\mathcal{M}$ over a Q-manifold $(\mathcal{M},Q)$ as odd vector fields on $\mathcal{E}^*$ of weight 1 preserving $\Gamma(\mathcal{E})$ and restricting to Q on $C^\infty(\mathcal{M})$. For example, for $\mathcal{M}=T[1]M$ with the de Rham differential, one obtains ordinary connections. It would be interesting to study if it is possible to define a notion of *Q-principal bundles* which unifies these ideas with those in [45] for L_∞-algebras.

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