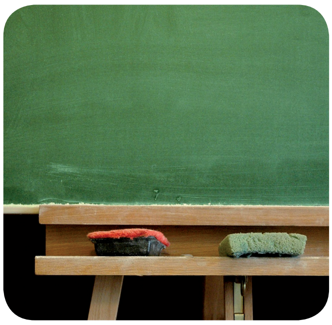
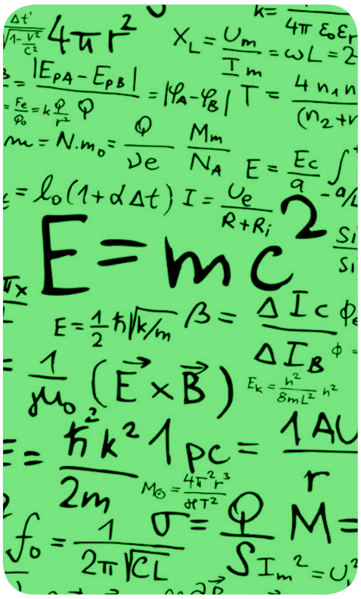
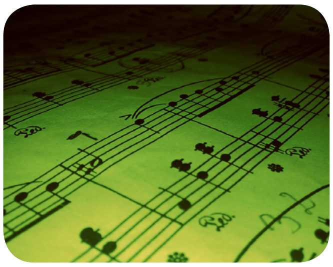

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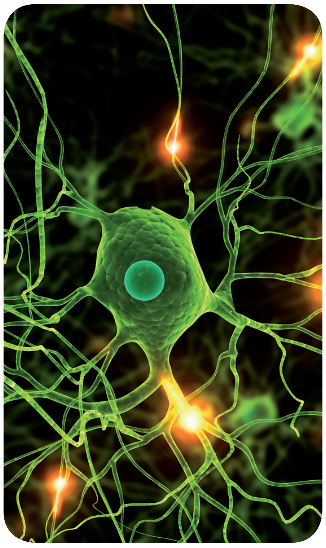
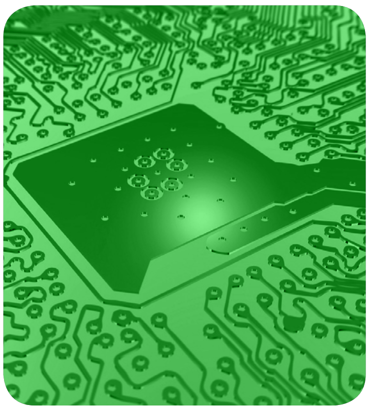
The pointwise convergence of the solution to the schrödinger equation

Daniel Eceizabarrena Pérez

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FINAL MASTER'S DISSERTATION

Master's Degree in Mathematics and Applications

THE POINTWISE CONVERGENCE OF THE SOLUTION TO THE SCHRÖDINGER EQUATION

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Introduction

The present document corresponds to the subject *Trabajo Fin de Máster* being part of the *Master's Degree in Mathematics and Applications* at the Universidad Autónoma de Madrid. The objective of this dissertation is to analyse the almost everywhere pointwise convergence of the solution to the Schrödinger equation to the given initial data.

As it is widely known, the **Schrödinger equation** is one of the pillars of quantum mechanics. It was first introduced by the Austrian physicist Erwin Schrödinger (Vienna, 1887-1961) in 1926 and models the evolution of the quantum state of a quantum system. The importance of this contribution earned him the Nobel Prize in Physics in 1933.

One of the most celebrated versions of the equation is the time-dependent one which is most generally given by

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\left[-\frac{\hbar^2}{2\mu}\Delta+V(x,t)\right]\Psi(x,t). \tag{0.1}$$

Here, i is the imaginary unit, $\hbar$ is Planck's constant and μ is the reduced mass of the particle. The equation involves the wave function of the quantum system here denoted by Ψ and the potential energy, V. By Δ we denote the Laplace operator. Nevertheless, it is usual to consider some extra assumptions which imply a simpler version of the equation. A case of interest is that of analysing the case of a free particle. This can be technically expressed by the suppression of the potential field, thus equation (0.1) becoming

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=-\frac{\hbar^2}{2\mu}\Delta\Psi(x,t).$$

and therefore

$$\frac{\partial}{\partial t}\Psi(x,t)=i\frac{\hbar}{2\mu}\Delta\Psi(x,t).$$

If we rename the constant $\frac{\hbar}{2\mu}=C$, we obtain which is probably the most basic and universal form of the Schrödinger equation given by

$$\frac{\partial}{\partial t}\Psi(x,t)=Ci\Delta\Psi(x,t). \tag{0.2}$$

For being an evolution equation, it is natural to assume certain initial data, which will represent the known state of a particular system, for example, in the present time. The objective is thus to know the future behaviour. Hence, if the problem is considered in the whole space $\mathbb{R}^n$, it can be stated as

$$\begin{cases} \frac{\partial}{\partial t}\Psi(x,t) - Ci\Delta_x\Psi(x,t) = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \Psi(x,0) = \Psi_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{0.3}$$

The problem we are to tackle is already visible in the given statement, for even if we were able to obtain an explicit form of the solution of (0.3), we would like, as we expect, to recover the known state when we go back to the starting time. Hence, we will focus on finding the properties the data should satisfy for the solution to converge to it pointwise almost everywhere when the time tends to zero.

This problem has been analysed since the 1980s, and it has become evident that the convenient spaces to work with are the Sobolev spaces H^s of fractional order. Precisely in 1980, in [4], Lennart Carleson treated the situation in one spacial dimension, successfully proving convergence for the case the exponent was $s \geq 1/4$. A year later, in 1981, Björn E. J. Dahlberg and Carlos E. Kenig were able to prove in [6] that the condition given by Carleson was sharp, showing the existence of functions in H^s with $s < 1/4$ for which convergence did not hold. The problem in $\mathbb{R}$ had therefore been solved.

The situation in higher dimensions has not been completely solved yet. Many authors such as Anthony Carbery in [3] and Michael Cowling in [5] achieved some positive results, and in 1987 Per Sjölin in [10] and Luis Vega in [13] showed independently that convergence holds if $s > 1/2$ no matter the dimension. More recent results have been obtained by Sanghyuk Lee in [7], who showed convergence in the two dimensional case $\mathbb{R}^2$ for $s > 3/8$, and Jean Bourgain in [2], who proved convergence for $s > 1/2 - 1/4n$ in $\mathbb{R}^n$ for $n \geq 3$. Bourgain also prove in [2] that it is necessary to ask $s \geq 1/2 - 1/n$ for convergence. Observe that when $n \geq 5$, this condition says $s > 1/4$, thus showing that the one-dimensional border cannot be achieved. The best known necessary condition in $\mathbb{R}^n$ with $n \geq 3$ is by Renato Lucà and Keith M. Rogers, who obtain that $s \geq 1/2 - 1/(n+2)$ is needed for convergence in [8].

In the following pages, we will analyse and prove several results mentioned above. Chapter 1 will be devoted to prove the characterisation for the one dimensional case. The following chapters will treat the higher dimensional case. In Chapter 2 we will work with general results applying in every dimension. More precisely, we will first prove the positive result of Sjölin and Vega based on [9], and we will also give the necessary condition by Lucà and Rogers. Finally, in Chapter 3 we will focus on the result of Lee, which is the best sufficient condition known so far for the two dimensional case.

Preliminaries

We will analyse the version

$$\begin{cases} u_t(x,t) - i\Delta_x u(x,t) = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{0.4}$$

of the Schrödinger equation. Note that by rescaling the initial value problem (0.2) can be reduced to (0.4). As we have said, it is not the most complete version of the equation, but it is probably the most extended form. Following the ideas presented in the introduction, the main objective is to examine the almost everywhere pointwise convergence of the solution to the initial data. But for that, it is completely necessary to have a concrete and explicit formula for the solution of (0.4). Fortunately, this is given by a well-known formula which can be obtained by means of the **Fourier transform**. We will make use of the definition

$$\mathscr{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi}\, dx$$

all along the present document. This way, we know that the Fourier transform is an isometry in L^2, as well as in the space of Schwartz or rapidly decreasing C^∞ functions $\mathscr{S}$, for which the inverse is given by

$$\mathscr{F}^{-1}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{2\pi i x\cdot\xi}\, dx.$$

Let us shortly sketch the steps to obtain a formula for the solution. We need to Fourier transform the equation only in the spacial variable, and by the linearity of the transform, it is enough to work out $\mathscr{F}(u_t)$ and $\mathscr{F}(\Delta u)$. Observe that if u is supposed to have enough regularity, since the transform variable and the differential variable are independent, we can write

$$\widehat{u_t}(\xi,t) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} u(x,t) e^{-2\pi i x\cdot\xi} dx = \frac{\partial}{\partial t}\int_{\mathbb{R}^n} u(x,t) e^{-2\pi i x\cdot\xi} dx = \frac{\partial}{\partial t}\widehat{u}(\xi,t).$$

On the other hand, a well-known property relating the Fourier transform and the derivatives says that $\mathscr{F}(D_i\varphi) = 2\pi i\xi_i\widehat{\varphi}$, from where we deduce that

$$\mathscr{F}_x(\Delta u) = \mathscr{F}_x(\sum_{j=1}^n D_j^2 u) = \sum_{j=1}^n \mathscr{F}_x(D_j^2 u) = -\sum_{j=1}^n 4\pi^2\xi_i^2\widehat{u} = -4\pi^2|\xi|^2\widehat{u}.$$

Therefore, the equation in (0.4) has become into an equation for $\widehat{u}$,

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -4\pi^2 i|\xi|^2\widehat{u},$$

which has a solution $\widehat{u}(\xi,t)=C(\xi)e^{-4\pi^2 it|\xi|^2}$. On the other hand, the initial condition in (0.4) forces $C(\xi)=\widehat{u_0}(\xi)$. Hence,

$$\widehat{u}(\xi,t)=\widehat{u_0}(\xi)e^{-4\pi^2 it|\xi|^2} \implies u(x,t)=\mathscr{F}_{\xi}^{-1}\left(\widehat{u_0}(\xi)e^{-4\pi^2 it|\xi|^2}\right).$$

Writing the inverse transform explicitly, we see that

$$u(x,t)=\int_{\mathbb{R}^n}\widehat{u_0}(\xi)e^{-4\pi^2 it|\xi|^2}e^{2\pi ix\cdot\xi}\,d\xi=\int_{\mathbb{R}^n}\widehat{u_0}(\xi)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}\,d\xi. \tag{0.5}$$

Definition 0.1. The expression at formula (0.5) is called **the solution to the Schrödinger's initial value problem** (0.4) and we denote it by $e^{it\Delta}u_0(x)$. Therefore,

$$e^{it\Delta}u_0(x)=\int_{\mathbb{R}^n}\widehat{u_0}(\xi)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}\,d\xi. \tag{0.6}$$

A property about the operator $e^{it\Delta}$ which is sometimes useful is that it is an isometry in L^2.

Proposition 0.2. *Let $t>0$. Then, the operator $e^{it\Delta}:L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is an isometry.*

Proof. By the construction we have seen above,

$$e^{it\Delta}f(x)=\mathscr{F}_{\xi}^{-1}\left(\widehat{f}(\xi)e^{-4\pi^2 it|\xi|^2}\right),$$

and since the Fourier transform is an isometry in L^2, we see that

$$\|e^{it\Delta}f\|_{L^2}=\|\widehat{f}(\xi)e^{-4\pi^2 it|\xi|^2}\|_{L^2}=\|\widehat{f}\|_{L^2}=\|f\|_{L^2}.$$

□

In our way to determining the properties for the solution to converge almost everywhere to the initial condition, that is to say, to satisfy

$$\lim_{t\to 0}e^{it\Delta}u_0(x)=u_0(x), \qquad \text{a.e.}, \tag{0.7}$$

we will need to work with several functional spaces. More precisely, we will have to decide in which spaces can the initial data u_0 lie if we want (0.7) to satisfy. One expects that no problems will arise when considering regular functions. For example, if we consider the extremely regular Schwartz functions, (0.7) can be easily checked. We write this first result here to have a first approach to the solution of the problem and also because it will be useful later in the setting of methods of approximation, since we know that the Schwartz space is a dense subspace in many functional spaces.

Proposition 0.3. *Let $u_0\in\mathscr{S}(\mathbb{R}^n)$ be a Schwartz function. Then,*

$$\lim_{t\to 0}e^{it\Delta}u_0(x)=u_0(x)$$

for almost every $x\in\mathbb{R}^n$.

Proof. It is a well-known result that the Fourier transform of a Schwartz function is again a Schwartz function. Now, considering the integral term in the solution (0.6), we see that there is a trivial bound given by

$$\left|\widehat{u_0}(\xi)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}\right| \leq |\widehat{u_0}(\xi)|,$$

which is integrable because $\mathscr{S} \subset L^1$. Hence, the dominated convergence theorem asserts that

$$\lim_{t\to 0} e^{it\Delta}u_0(x) = \int_{\mathbb{R}^n} \widehat{u_0}(\xi)e^{2\pi i x\cdot\xi}\, d\xi = u_0(x),$$

which is given by the Fourier inversion formula. □

When we consider not so regular functions, it turns out that the adequate spaces are those we call fractional Sobolev spaces or simply Sobolev spaces. Recall the definition for usual Sobolev spaces, for which we will use the multi-index notation.

Definition 0.4. For $k \in \mathbb{N}$, the **Sobolev space** $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ is defined to be

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid D^\alpha f \in L^2(\mathbb{R}^n), \quad \forall |\alpha| \leq k\}.$$

In other words, it is the space of L^2 functions whose derivatives up to order k are also in L^2.

As we have said above, the Fourier transform is an isometry in $L^2(\mathbb{R}^n)$, so that means that for $f \in H^k(\mathbb{R}^n)$, since its derivatives are in L^2, we have that $\mathscr{F}(D^\alpha f) \in L^2(\mathbb{R}^n)$. By the properties of the derivatives of the Fourier transform, and using the standard multi-index notation, $\mathscr{F}(D^\alpha f) = (2\pi i\xi)^\alpha \widehat{f}$. Hence,

$$D^\alpha f \in L^2(\mathbb{R}^n) \Leftrightarrow \xi^\alpha \widehat{f} \in L^2(\mathbb{R}^n).$$

This can also be equivalently written as $|\xi|^{|\alpha|}\widehat{f} \in L^2(\mathbb{R}^n)$. Therefore, by writing these conditions in their integral forms and since $\widehat{f} \in L^2(\mathbb{R}^n)$, we can write

$$H^k(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} (1+|\xi|^{2k})|\widehat{f}(\xi)|^2\, d\xi < \infty \right\}. \tag{0.8}$$

These spaces can also be equivalently presented as

$$\begin{aligned} H^k(\mathbb{R}^n) &= \left\{ f \in L^2(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} (1+|\xi|)^{2k}|\widehat{f}(\xi)|^2\, d\xi < \infty \right\} \\ &= \left\{ f \in L^2(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} (1+|\xi|^2)^k|\widehat{f}(\xi)|^2\, d\xi < \infty \right\}. \end{aligned} \tag{0.9}$$

The main advantage of working with these alternative expressions rather that with the classical ones is that k need not be natural now. Indeed, the integrals we write make sense for every $k > 0$, which precisely allows us to define the fractional Sobolev spaces.

Definition 0.5. Let $s > 0$. Then, the **fractional order Sobolev space** H^s is defined as

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty \right\},$$

or equivalently as any of the forms in (0.8) and (0.9). The norm we will use is given by

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi.$$

As we have suggested before, the result of Proposition 0.3 will play a key role. This is because the space of Schwartz functions is not only dense in L^2, as we already know, but it is also dense in the fractional Sobolev spaces.

Proposition 0.6. *The space of Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every $s \geq 0$.*

Proof. We will do it in two steps. First, we will see that the space of L^2 functions with compact Fourier support (this is to say, that their Fourier transforms have compact support) is dense in H^s, and after that, we will check the density of $\mathscr{S}$ in the latter.

Let $f \in H^s$ so that $\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty$. The convergence of the integral shows that for any $\epsilon > 0$ there exists $M > 0$ such that

$$\int_{|\xi|>M} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \epsilon/2.$$

Hence, if we choose $\widehat{\varphi}(\xi) = \widehat{f}(\xi)\chi_{B(0,M)}(\xi)$, we see that $\widehat{\varphi}$ has compact support and that it is in L^2 because $f \in L^2$ and thus $\widehat{f} \in L^2$ too. Moreover,

$$\|f-\varphi\|^2_{H^s} = \int_{\mathbb{R}^n} |\widehat{f}(\xi) - \widehat{\varphi}(\xi)|^2 (1+|\xi|^2)^s \, d\xi = \int_{|\xi|>M} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \epsilon/2.$$

Now since $\mathscr{S}$ is dense in L^2, and since $\widehat{\varphi} \in L^2$, there exists $\widehat{h} \in \mathscr{S}$ such that

$$\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi < \frac{\epsilon}{2(1+4M^2)^s}.$$

Consider a cutoff function ϕ_M supported in $B(0,2M)$ and being 1 in $B(0,M)$ and define $\widehat{h_M} = \widehat{h}\phi_M \in \mathscr{S}$. Then, denoting the annulus of radii M and $2M$ as $A(M,2M)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi) - \widehat{h_M}(\xi)|^2 \, d\xi &= \int_{B(0,M)} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi + \int_{A(M,2M)} |\widehat{h_M}(\xi)|^2 \, d\xi \\
&\leq \int_{B(0,M)} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi + \int_{A(M,2M)} |\widehat{h}(\xi)|^2 \, d\xi \\
&= \int_{B(0,M)} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi + \int_{A(M,2M)} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi) - \widehat{h}(\xi)|^2 \, d\xi < \frac{\epsilon}{2(1+4M^2)^s}.
\end{aligned}$$

Hence,

$$\|\varphi - h_M\|^2_{H^s} = \int_{B(0,2M)} |\widehat{\varphi}(\xi) - \widehat{h_M}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \frac{\epsilon}{2},$$

since $|\xi|^2 \leq 4M^2$. By joining the two previous results, by the triangle inequality we have

$$\|f - h_M\|^2_{H^s} < \|f - \varphi\|^2_{H^s} + \|\varphi - h_M\|^2_{H^s} < \epsilon.$$

□

Therefore, since we can approximate Sobolev functions by means of Schwartz' rapidly decreasing functions, the problem will reduce many times, as we will see, to proving some maximal a priori estimate in terms of the Sobolev norm. This is because we will be able to split any $f \in H^s$ into $\varphi \in \mathscr{S}$ and $g = f - \varphi \in H^s$ so that φ is regular and g has a norm as small as we need.

1

Analysis in One Dimension

This chapter is devoted to analyse the results in one dimension. As we have said, the issue of convergence is completely determined in the case of $\mathbb{R}$, so we will be able to characterise the convergence property (0.7) in terms of the exponent of the Sobolev spaces defined in Definition 0.5. We will split the chapter in two sections. First, we will analyse the positive result, and we will see counterexamples afterwards which will determine the necessary condition.

1.1 The Positive Result

The first positive result regarding the problem of convergence to the initial data was obtained by Lennart Carleson in [4]. We present his result in the following theorem.

Theorem 1.1. *Let $s \geq \frac{1}{4}$ and consider $f \in H^s(\mathbb{R})$. Then,*

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x)$$

almost everywhere.

This section is devoted to proving Theorem 1.1, for which we will need to take several steps. The keystone, as suggested, will be to estimate a maximal function of the solution $e^{it\Delta} f$ by means of the Sobolev norm $\|f\|_{H^{1/4}(\mathbb{R})}$. But before we do so, we need to present an auxiliary lemma which is also due to Carleson [4].

Lemma 1.2. *Let $a, b \in (-2,2)$ and $\alpha \in (0,1)$. Then,*

$$\left| \int_{\mathbb{R}} e^{i(a\xi + b\xi^2)} \frac{d\xi}{|\xi|^\alpha} \right| \leq C_\alpha \left(|b|^{\alpha - 1/2} |a|^{-\alpha} + |a|^{\alpha - 1} \right). \tag{1.1}$$

We postpone the proof of Lemma 1.2 to the end of the present section for being quite technical.

As we have said in the beginning of the section, the main result we need to prove Theorem 1.1 is a maximal estimate for the solution $e^{it\Delta}f$ by means of the Sobolev norm $\|f\|_{H^{1/4}(\mathbb{R})}$. We present it in the following proposition.

Proposition 1.3. *Let $f \in \mathscr{S}(\mathbb{R})$ any Schwartz function. Then, there exists a constant $C > 0$ such that*

$$\|\sup_{t>0}|e^{it\Delta}f|\|_{L^4(\mathbb{R})} \leq C\|f\|_{H^{1/4}(\mathbb{R})}.$$

Proof. For technical reasons which will be clear in the following lines, we will prove the estimate in a ball $B(0,R)$ with $R > 0$. In other words, we will first prove

$$\|\sup_{t>0}|e^{it\Delta}f|\|_{L^4(B(0,R))} \leq C\|f\|_{H^{1/4}(\mathbb{R})}, \tag{1.2}$$

C being a constant independent of R. Indeed, this is enough to prove the estimate in the statement, because if (1.2) holds for every $R > 0$, then by taking the limit $R \to \infty$, we see that

$$\lim_{R\to\infty}\|\sup_{t>0}|e^{it\Delta}f|\|_{L^4(B(0,R))} = \|\sup_{t>0}|e^{it\Delta}f|\|_{L^4(\mathbb{R})}$$

by the monotone convergence theorem. So let us prove (1.2).

Working with the norm of a supremum is not convenient in general, so we will remove it. Fix $x \in \mathbb{R}$. Then, there exists $t(x) > 0$, which generated a measurable function, such that $|e^{it(x)\Delta}f(x)| \geq \frac{1}{2}\sup_{t>0}|e^{it\Delta}f(x)|$. In other words,

$$\sup_{t>0}|e^{it\Delta}f(x)| \leq 2|e^{it(x)\Delta}f(x)|. \tag{1.3}$$

Then, it will be enough to obtain an estimate for the L^4 norm of $e^{it(x)\Delta}f(x)$, which is now a function only of x. Recall that we can argue about the L^p norms by duality. Indeed, for any $f \in L^p$, there exists a function $w \in L^{p'}$ such that $\|f\|_{L^p} = \int fw$. Moreover, this function w has unitary norm. Hence, there exists $w \in L^{4/3}(B(0,R))$ with $\|w\|_{L^{4/3}} = 1$ such that

$$\|e^{it(\cdot)\Delta}f(\cdot)\|_{L^4(B(0,R))} = \int_{\mathbb{R}} e^{it(x)\Delta}f(x)w(x)\,dx,$$

since w can be considered to be supported in $B(0,R)$. If we were able to prove that for functions $w \in L^{4/3}(B(0,R))$ supported in $B(0,R)$

$$\int_{\mathbb{R}} e^{it(x)\Delta}f(x)w(x)\,dx \leq C\|f\|_{H^{1/4}}\|w\|_{L^{4/3}} \tag{1.4}$$

with some constant $C > 0$ independent of the functions and of R, then by (1.3),

$$\begin{aligned}\|\sup_{t>0}e^{it\Delta}f\|_{L^4(B(0,R))} &\leq 2\|e^{it(x)\Delta}f(x)\|_{L^4(B(0,R))} = 2\int_{\mathbb{R}} e^{it(x)\Delta}f(x)w(x)\,dx\\ &\leq 2C\|f\|_{H^{1/4}}\|w\|_{L^{4/3}} = 2C\|f\|_{H^{1/4}}\end{aligned}$$

because w is unitary, and we would obtain (1.2). Hence, let us prove (1.4). Square the integral and write the definition of the solution to get

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi i(x\xi-2\pi t(x)\xi^2)}\,d\xi w(x)\,dx\right|^2.$$

Observe that $w \in L^{4/3}(B(0,R)) \subset L^1(B(0,R))$, so Fubini's theorem allows us to change the order of integration, and if we write

$$\left|\int_{\mathbb{R}} \widehat{f}(\xi)|\xi|^{1/4}\left(\int_{\mathbb{R}} e^{2\pi i(x\xi-2\pi t(x)\xi^2)}\frac{w(x)}{|\xi|^{1/4}}\,dx\right)d\xi\right|^2,$$

we apply Cauchy-Schwarz' inequality to obtain a bound by

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2|\xi|^{1/2}\,d\xi \int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{2\pi i(x\xi-2\pi t(x)\xi^2)}\frac{w(x)}{|\xi|^{1/4}}\,dx\right|^2 d\xi. \tag{1.5}$$

The left-hand side integral at (1.5) can be bounded by

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2|\xi|^{1/2}\,d\xi \le \int_{\mathbb{R}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{1/4}\,d\xi = \|f\|^2_{H^{1/4}},$$

so we obtain one of the desired values. We need to deal with the right-hand side integral of (1.5). Use $|z|^2 = z\overline{z}$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{2\pi i(x\xi-2\pi t(x)\xi^2)}\frac{w(x)}{|\xi|^{1/4}}\,dx\right|^2 d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi i\xi(x-y)-4\pi^2 i\xi^2(t(x)^2-t(y)^2)}w(x)\overline{w(y)}\,dx\,dy\frac{d\xi}{|\xi|^{1/2}}.\end{aligned} \tag{1.6}$$

Now, we want to make use of Lemma 1.2. Observe that for $\alpha = 1/2$, the term b disappears and

$$\left|\int_{\mathbb{R}} e^{i(a\xi+b\xi^2)}\frac{d\xi}{|\xi|^{\alpha}}\right| \le C|a|^{-1/2}.$$

But for that, we need to change the order of integration. In principle, we cannot do it since $|\xi|^{-1/2}$ is not integrable. But observe that Fatou's lemma allows us to work with the $L \to \infty$ limit of the integral in $(-L,L)$ in variable ξ. In this situation, $|\xi|^{-1/2}$ is integrable and hence Fubini's theorem allows us to change the order of the integral in (1.6). Moreover, the proof of Lemma 1.2 shows that its conclusions are also valid in intervals $(-L,L)$, so we see that if $a = 2\pi(x-y)$ and $b = -4\pi^2(t(x)^2 - t(y)^2)$, we can say that with absolute values,

$$(1.6) \le C\int_{\mathbb{R}^2} \frac{|w(x)||w(y)|}{|x-y|^{1/2}}\,dx\,dy.$$

Now Hölder's inequality gives us the way to write

$$C\int_{\mathbb{R}^2} \frac{|w(x)||w(y)|}{|x-y|^{1/2}}\,dx\,dy \le \|w\|_{L^{4/3}}\left\|\int_{\mathbb{R}} \frac{|w(x)|}{|x-y|^{1/2}}\,dx\right\|_{L^4}.$$

Observe that the L^4 norm can be solved by the Hardy-Littlewood-Sobolev inequality which we present here.

Proposition (Hardy-Littlewood-Sobolev Inequality)**.** *Consider $f \in L^p(\mathbb{R}^n)$ and indices $0 < \gamma < n$, $1 < p < q < \infty$ such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}.$$

Then,

$$\|f * |y|^{-\gamma}\|_{L^q(\mathbb{R}^n)} \leq A_{p,q}\|f\|_{L^p(\mathbb{R}^n)}.$$

Its proof has not a big interest here and hence we refer the reader to Proposition B.1 at Appendix B.1. Let us use it. We have $\gamma = 1/2$ and $q = 4$, so

$$\frac{1}{p} = \frac{1}{q} + \frac{n-\gamma}{n} = \frac{1}{4} + 1 - \frac{1}{2} = \frac{3}{4}$$

and $p = 4/3$. Therefore, the Hardy-Littlewood-Sobolev inequality asserts that

$$\left\| \int_{\mathbb{R}} \frac{|w(y)|}{|x-y|^{1/2}}\, dy \right\|_{L^4} \leq C\|w\|_{L^{4/3}},$$

and (1.6) $\leq C\|w\|^2_{L^{4/3}}$. Hence, we have obtained

$$\left| \int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x)\, dx \right|^2 \leq C\|f\|^2_{H^{1/4}}\|w\|^2_{L^{4/3}},$$

from which we deduce (1.4). □

Remark 1.4. Observe that

$$\|f\|_{H^{1/4}} = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{1/4}\, d\xi \leq \int_{\mathbb{R}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{1/4+\epsilon}\, d\xi = \|f\|_{H^{1/4+\epsilon}},$$

for any $\epsilon > 0$. Hence, Proposition 1.3 also shows that

$$\|\sup_{t>0} \left| e^{it\Delta} f \right| \|_{L^4(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})},$$

for every $s \geq 1/4$.

The estimate in Proposition 1.3 and more generally in Remark 1.4 will be the main property we are going to exploit in order to obtain convergence to the initial data. But for that, we need to extend it to general functions in $H^{1/4}$. The fact that Schwartz functions are dense in H^s as we saw in Proposition 0.6 leads the way to the extension. Thus, we will develop an argument of density.

Corollary 1.5. *There exists a constant $C > 0$ such that*

$$\|\sup_{t>0} \left| e^{it\Delta} f \right| \|_{L^4(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}, \tag{1.7}$$

for every $f \in H^s(\mathbb{R})$ with $s \geq 1/4$.

Proof. Remark 1.4 shows that it is enough to prove the estimate for $s = 1/4$, since $H^s \subset H^{1/4}$ with $\|f\|_{H^{1/4}} \le \|f\|_{H^s}$. So let $f \in H^{1/4}$. By the density of Schwartz functions, consider a sequence $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{S}$ such that $f_m \to f$ in $H^{1/4}$. As we have seen in Proposition 1.3, for every $m \in \mathbb{N}$,

$$\|e^{it\Delta} f_m\|_{L^4(\mathbb{R})} \le \|\sup_{t>0} |e^{it\Delta} f_m|\|_{L^4(\mathbb{R})} \le C\|f_m\|_{H^{1/4}(\mathbb{R})} < \infty, \tag{1.8}$$

so $e^{it\Delta} f_m \in L^4$. Notice that taking limits in (1.8), we would like to prove that $\lim_{m\to\infty} \|e^{it\Delta} f_m\|_{L^4(\mathbb{R})} = \|e^{it\Delta} f\|_{L^4(\mathbb{R})}$. Now observe that by the linearity of the operator $e^{it\Delta}$, we can write

$$\|e^{it\Delta} f_m - e^{it\Delta} f_l\|_{L^4} = \|e^{it\Delta}(f_m - f_l)\|_{L^4} \le C\|f_m - f_l\|_{H^{1/4}} \to 0$$

when $m, l \to \infty$ because $\{f_m\}_{m\in\mathbb{N}}$ is a convergent sequence and thus a Cauchy sequence in $H^{1/4}$. As a consequence, $\{e^{it\Delta} f_m\}$ is a Cauchy sequence in L^4 and hence convergent. We want to see that the limit is precisely $e^{it\Delta} f$. For that, observe that

$$e^{it\Delta} f(x) = \mathscr{F}^{-1}\left(\widehat{f}(\xi) e^{-4\pi^2 t|\xi|^2}\right). \tag{1.9}$$

Since $f_m \to f$ in $H^{1/4} \subset L^2$, then by definition of the Fourier transform in L^2, we have $\widehat{f_m} \to \widehat{f}$ in L^2. But of course, the exponentials are unitary, so

$$e^{-4\pi^2 t|\xi|^2} \widehat{f_m} \to e^{-4\pi^2 t|\xi|^2} \widehat{f} \implies \mathscr{F}^{-1}\left(\widehat{f_m}(\xi) e^{-4\pi^2 t|\xi|^2}\right) \to \mathscr{F}^{-1}\left(\widehat{f}(\xi) e^{-4\pi^2 t|\xi|^2}\right)$$

in L^2 because the Fourier transform and its inverse are continuous operators. Hence, this shows by (1.9) that

$$\{e^{it\Delta} f_m\} \to e^{it\Delta} f \quad \text{in } L^2.$$

But we are working in L^4, not in L^2. Nevertheless, if g were the L^4 limit, by convergence in L^2 we know that there exists a subsequence $\{e^{it\Delta} f_{m_k}\}_k \to e^{it\Delta} f$ almost everywhere, and by convergence in L^4 a subsequence $\{e^{it\Delta} f_{m_{k_l}}\}_l$ converges almost everywhere to g. Since two almost everywhere limits must be the same, $g = e^{it\Delta} f$ and

$$\{e^{it\Delta} f_m\} \to e^{it\Delta} f \quad \text{in } L^4$$

as we wished. This implies by (1.8) that

$$\|e^{it\Delta} f\|_{L^4} = \lim_{m\to\infty} \|e^{it\Delta} f_m\|_{L^4} \le C \lim_{m\to\infty} \|f_m\|_{H^{1/4}} = C\|f\|_{H^{1/4}}, \tag{1.10}$$

which is a kind of partial result for (1.7). Now we need to argue with the supremum. Again, the objective is to prove convergence

$$\|\sup_{t>0} |e^{it\Delta} f|\|_{L^4} = \lim_{m\to\infty} \|\sup_{t>0} |e^{it\Delta} f_m|\|_{L^4} \tag{1.11}$$

in order to apply the estimate in $H^{1/4}$, but in this case linearity is not available as before. Anyway, we can write

$$\begin{aligned} \sup_{t>0} |e^{it\Delta}(f_m - f_l)| &= \sup_{t>0} |e^{it\Delta} f_m - e^{it\Delta} f_l| \ge \sup_{t>0} \left| |e^{it\Delta} f_m| - |e^{it\Delta} f_l| \right| \\ &\ge \left| \sup_{t>0} |e^{it\Delta} f_m| - \sup_{t>0} |e^{it\Delta} f_l| \right|, \end{aligned} \tag{1.12}$$

where the first inequality is the triangle inequality and the second one is true because for functions $g, h \geq 0$,

$$|\sup g - \sup h| \leq \sup |g - h|.$$

Hence, from (1.12) we can write

$$\|\sup_{t>0} \left|e^{it\Delta} f_m\right| - \sup_{t>0} \left|e^{it\Delta} f_l\right| \|_{L^4} \leq \|\sup_{t>0} \left|e^{it\Delta}(f_m - f_l)\right| \|_{L^4} \leq C \|f_m - f_l\|_{H^{1/4}} \to 0$$

when $m, l \to \infty$. Then, $\{\sup_{t>0} \left|e^{it\Delta} f_m\right|\}_m$ is Cauchy and therefore it has a limit, say g, in L^4. We want to compare that limit with $\sup_{t>0} \left|e^{it\Delta} f\right|$ as suggested in (1.11). Indeed,

- We can consider a subsequence $\{\sup_{t>0} \left|e^{it\Delta} f_{m_k}\right|\}_k$ which converges pointwise to g.
- By the convergence $\{e^{it\Delta} f_m\} \to e^{it\Delta} f$ in L^4, we consider a sub-subsequence $\{m_{k_l}\}_l$ so that $\{e^{it\Delta} f_{m_{k_l}}\}_l \to e^{it\Delta} f$ pointwise.

Hence,

$$g(x) = \lim_{l\to\infty} \sup_{t>0} \left|e^{it\Delta} f_{m_{k_l}}(x)\right| \geq \lim_{l\to\infty} \left|e^{it\Delta} f_{m_{k_l}}(x)\right| = \left|e^{it\Delta} f(x)\right|$$

almost everywhere for every $t > 0$, and thus $g(x) \geq \sup_{t>0} \left|e^{it\Delta} f(x)\right|$. This is not precisely (1.11), but it is sufficient for our objective, since we have proven that

$$\|\sup_{t>0} \left|e^{it\Delta} f\right| \|_{L^4} \leq \|g\|_{L^4} = \lim_{m\to\infty} \|\sup_{t>0} \left|e^{it\Delta} f_m\right| \|_{L^4}.$$

As a consequence,

$$\|\sup_{t>0} \left|e^{it\Delta} f\right| \|_{L^4} \leq \lim_{m\to\infty} \|\sup_{t>0} \left|e^{it\Delta} f_m\right| \|_{L^4} \leq C \lim_{m\to\infty} \|f_m\|_{H^{1/4}} = C\|f\|_{H^{1/4}},$$

and we are done. □

Once we have been able to prove the estimate for functions in H^s for any $s \geq 1/4$, we are ready to give the main result. The technique is similar to the standard proof of Lebesgue's differentiation theorem.

Proof of Theorem 1.1. The result is again given by an argument of density. Consider a function $f \in H^s$, and since $\mathscr{S}$ is dense in H^s, for $\epsilon > 0$ let us consider $\varphi \in \mathscr{S}$ such that $\|f - \varphi\|_{H^s} \leq \epsilon$. If we call $g = f - \varphi$, observe that we have decomposed $f = g + \varphi$ into two parts, one being regular and the other one having small norm. Observe first that the issue of convergence for Schwartz data is trivially solved as we saw in Proposition 0.3, so the problem is basically translated to functions in H^s which have norm as small as we wish.

Recall that we want to check that $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere. This is equivalent to proving $\lim_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| = 0$, or in other words,

$$m\left(\left\{x \mid \limsup_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| \neq 0\right\}\right) = 0.$$

We write the upper limit to avoid problems of existence. In any case, it is also clear that

$$\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| \neq 0\right\} = \bigcup_{k\in\mathbb{N}}\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| > \frac{1}{k}\right\},$$

from which we deduce that

$$\begin{aligned} &m\left(\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| \neq 0\right\}\right) \\ &\quad\leq \sum_{k\in\mathbb{N}} m\left(\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| > \frac{1}{k}\right\}\right). \end{aligned} \tag{1.13}$$

Therefore, it is enough to see that every summand in (1.13) is null. As suggested before, let us translate the problem from f to g. Indeed,

$$\begin{aligned} \left|e^{it\Delta}f(x)-f(x)\right| &= \left|e^{it\Delta}g(x)-g(x)+e^{it\Delta}\varphi(x)-\varphi(x)\right| \\ &\leq \left|e^{it\Delta}g(x)-g(x)\right| + \left|e^{it\Delta}\varphi(x)-\varphi(x)\right|, \end{aligned}$$

and taking limits $t \to 0$, by the convergence result for Schwartz functions analysed in Proposition 0.3, we get

$$\limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| \leq \limsup_{t\to 0}\left|e^{it\Delta}g(x)-g(x)\right|,$$

so for all $k \in \mathbb{N}$, we have

$$\begin{aligned} &m\left(\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}f(x)-f(x)\right| > \frac{1}{k}\right\}\right) \\ &\quad\leq m\left(\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}g(x)-g(x)\right| > \frac{1}{k}\right\}\right). \end{aligned}$$

Now, observe that $\left|e^{it\Delta}g(x)-g(x)\right| \leq \left|e^{it\Delta}g(x)\right| + |g(x)|$, so by taking limits we see that

$$\limsup_{t\to 0}\left|e^{it\Delta}g(x)-g(x)\right| \leq \limsup_{t\to 0}\left|e^{it\Delta}g(x)\right| + |g(x)| \leq \sup_{t>0}\left|e^{it\Delta}g(x)\right| + |g(x)|.$$

Also since $\sup_{t>0}\left|e^{it\Delta}g(x)\right|+|g(x)| > 1/k$ implies that either $\sup_{t>0}\left|e^{it\Delta}g(x)\right| > 1/2k$ or $|g(x)| > 1/2k$, we can write

$$\begin{aligned} &m\left(\left\{x \mid \limsup_{t\to 0}\left|e^{it\Delta}g(x)-g(x)\right| > \frac{1}{k}\right\}\right) \\ &\quad\leq m\left(\left\{x \mid \sup_{t>0}\left|e^{it\Delta}g(x)\right| > \frac{1}{2k}\right\}\right) + m\left(\left\{x \mid |g(x)| > \frac{1}{2k}\right\}\right). \end{aligned} \tag{1.14}$$

Let us treat each of the terms in (1.14) separately.

- On the one hand, Chebyshev's inequality and the result in Corollary 1.7 show that

$$m\left(\left\{x \mid \sup_{t>0}\left|e^{it\Delta}g(x)\right| > \frac{1}{2k}\right\}\right) \leq (2k)^4 \|\sup_{t>0}\left|e^{it\Delta}g\right|\|_{L^4}^4 \leq C(2k)^4\|g\|_{H^s}^4.$$

- On the other hand, by Chebyshev and Plancherel (because $g \in H^{1/4} \subset L^2$),

$$m\left(\left\{x \mid |g(x)| > \frac{1}{2k}\right\}\right) \leq (2k)^2 \|g\|_{L^2}^2 = (2k)^2 \int_{\mathbb{R}} |\widehat{g}(\xi)|^2 \, d\xi$$
$$\leq (2k)^2 \int_{\mathbb{R}} |\widehat{g}(\xi)|^2 (1+|\xi|^2)^s \, d\xi = (2k)^2 \|g\|_{H^s}^2.$$

Therefore,

$$(1.14) \leq (2k)^2 \|g\|_{H^s}^2 \left(C(2k)^2 \|g\|_{H^s}^2 + 1\right) < (2k)^2 \epsilon^2 \left(C(2k)^2 \epsilon^2 + 1\right),$$

which, remember, is valid for every $\epsilon > 0$. Hence, letting $\epsilon \to 0$ and going back to (1.13), we see that for every $k \in \mathbb{N}$,

$$m\left(\left\{x \mid \limsup_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| \neq 0\right\}\right) = 0,$$

which completes the proof. □

To complete this section, we will give the proof of Lemma 1.2 which was postponed in the beginning of the section.

Proof of Lemma 1.2. We will need to use tools regarding oscillatory integrals developed in Appendix A, so we will keep the notation introduced there. We first assume $b = 0$. Then, we are looking for the proof of

$$\left|\int_{\mathbb{R}} e^{iat} \frac{dt}{|t|^\alpha}\right| \leq C_\alpha |a|^{\alpha-1}. \tag{1.15}$$

Observe that the left-hand side of (1.15) is an oscillatory integral with parameter a, with phase $\phi(t) = t$ and $\psi(t) = |t|^{-\alpha}$. We split the integral into three. For $\epsilon > 0$, we will consider the integrals in $(-\infty, -\epsilon), (-\epsilon, \epsilon)$ and (ϵ, ∞). In the middle interval,

$$\left|\int_{-\epsilon}^{\epsilon} e^{iat} \frac{dt}{|t|^\alpha}\right| \leq \int_{-\epsilon}^{\epsilon} \frac{dt}{|t|^\alpha} = 2\int_0^{\epsilon} t^{-\alpha}\, dt = \frac{2}{1-\alpha}\epsilon^{1-\alpha}.$$

Choose $\epsilon = 1/|a|$ so that $\epsilon^{1-\alpha} = |a|^{\alpha-1}$ and the bound is

$$\left|\int_{-\epsilon}^{\epsilon} e^{iat} \frac{dt}{|t|^\alpha}\right| \leq \frac{2}{1-\alpha}|a|^{\alpha-1}.$$

Let us consider the remaining integrals now. We will need to use the results of Van der Corput, and more precisely Corollary A.9. With ϕ and ψ already defined, and since $|\psi'(t)| = \alpha |t|^{-\alpha-1}$, $|\phi'(t)| \geq 1$, we have

$$\left|\int_{1/|a|}^{R} e^{iat} \frac{dt}{|t|^\alpha}\right| \leq C|a|^{-1}\left(\psi(R) + \int_{1/|a|}^{R} |\psi'(t)|\, dt\right) = C|a|^{-1}\left(R^{-\alpha} - t^{-\alpha}\Big|_{1/|a|}^{R}\right)$$
$$= C|a|^{-1}|a|^{\alpha} = C|a|^{\alpha-1},$$

which is valid for any $R > 1/|a|$. On the other hand,

$$\left|\int_{-R}^{-1/|a|} e^{iat} \frac{dt}{|t|^\alpha}\right| = \left|\int_{1/|a|}^{R} e^{-iat} \frac{dt}{|t|^\alpha}\right|,$$

and the situation is the same excepting that the phase is $\phi(t) = -t$. But as before, $|\phi'(t)| = 1$ and the estimates of Van der Corput are allowed, hence the result is the same, again for any $R > 1/|a|$. Therefore, we obtain

$$\int_{-R}^{R} e^{iat} \frac{dt}{|t|^\alpha} \leq \left(\frac{2}{1-\alpha} + 2C \right) |a|^{\alpha-1}, \quad \forall R > 1/|a|,$$

so (1.15) is satisfied by letting $R \to \infty$.

We can thus assume $b \neq 0$. By the absolute values in (1.1), we may even assume $b > 0$. Let us simplify (1.1). Change variables $t = b^{1/2}\xi$ in the integral so that

$$\int_{\mathbb{R}} e^{i(a\xi + b\xi^2)} \frac{d\xi}{|\xi|^\alpha} = \int_{\mathbb{R}} e^{i(ab^{-1/2}t + t^2)} \frac{b^{-1/2}}{|t|^\alpha b^{-\alpha/2}} dt = b^{(\alpha-1)/2} \int_{\mathbb{R}} e^{i(2At+t^2)} \frac{dt}{|t|^\alpha},$$

if we call $2A = ab^{-1/2}$. After this change of variables, (1.1) is equivalent to proving

$$\begin{aligned} \int_{\mathbb{R}} e^{i(2At+t^2)} \frac{dt}{|t|^\alpha} &\leq b^{(1-\alpha)/2} C_\alpha \left(b^{\alpha - 1/2} |a|^{-\alpha} + |a|^{\alpha-1} \right) \\ &= C_\alpha \left(b^{\alpha/2} |a|^{-\alpha} + b^{(1-\alpha)/2} |a|^{\alpha-1} \right) \\ &= C_\alpha \left(|2A|^{-\alpha} + |2A|^{\alpha-1} \right) = K_\alpha \left(|A|^{-\alpha} + |A|^{\alpha-1} \right). \end{aligned} \tag{1.16}$$

Consider two different cases for A. Let $|A| \leq 2$. Split the integral as in case $b = 0$, and for the same reason,

$$\left| \int_{-\epsilon}^{\epsilon} e^{i(2At+t^2)} \frac{dt}{|t|^\alpha} \right| \leq \frac{2}{1-\alpha} \epsilon^{1-\alpha}.$$

Let us focus on (ϵ, R), for $R > \epsilon$. If we now take $\phi(t) = 2At + t^2$, then $|\phi''(t)| = 2$, and what is more, if we fix $\phi^* = \phi/|A|$, then $|(\phi^*)''(t)| = 2/|A| \geq 1$, so with $\psi(t) = t^{-\alpha}$ as before, we can use the second order Van der Corput estimate from Corollary A.9 to say that

$$\left| \int_{\epsilon}^{R} e^{i\phi(t)} \frac{dt}{t^\alpha} \right| = \left| \int_{\epsilon}^{R} e^{i|A|\phi^*(t)} \frac{dt}{t^\alpha} \right| \leq C|A|^{-1/2} \left(R^{-\alpha} - t^{-\alpha} \Big|_{\epsilon}^{R} \right) = C|A|^{-1/2} \epsilon^{-\alpha}.$$

On the other hand,

$$\left| \int_{-R}^{-\epsilon} e^{i(2At+t^2)} \frac{dt}{|t|^\alpha} \right| = \left| \int_{\epsilon}^{R} e^{i(-2At+t^2)} \frac{dt}{t^\alpha} \right|,$$

so considering now $\phi(t) = -2At + t^2$, the second derivative does not change with respect to the previous case, so if we repeat the process with the idea of ϕ^*, we obtain exactly the same bound. Hence, for every $R > \epsilon$,

$$\left| \int_{-R}^{R} e^{i(2At+t^2)} \frac{dt}{|t|^\alpha} \right| \leq \frac{2}{1-\alpha} \epsilon^{1-\alpha} + 2C|A|^{-1/2} \epsilon^{-\alpha}.$$

If we consider $\epsilon = 1/|A|$ as before, the bound we obtain is $C_\alpha(|A|^{\alpha-1} + |A|^{\alpha-1/2})$. We wish to have a bound with $|A|^{\alpha-1}$. Observe that if $|A| \leq 1$, then $|A| \leq |A|^{1/2}$. On the other hand, if $1 \leq |A| \leq 2$, then $|A| \leq 2 \leq 2|A|^{1/2}$, so in any case $|A| \leq 2|A|^{1/2}$. Hence, $|A|^{\alpha-1/2} \leq 2|A|^{\alpha-1}$, and hence we obtain a bound $C_\alpha |A|^{\alpha-1}$, which obviously satisfies (1.16).

Hence we are only left with the case $|A|>2$. In this case, we need to split the integral into more pieces:

$$\begin{aligned}\int_{\mathbb{R}} &= \int_{-\infty}^{-2|A|} + \int_{-2|A|}^{-|A|/2} + \int_{-|A|/2}^{-1/|A|} + \int_{-1/|A|}^{1/|A|} + \int_{1/|A|}^{|A|/2} + \int_{|A|/2}^{2|A|} + \int_{2|A|}^{\infty} \\ &= I_3^- + I_2^- + I_1^- + I_0 + I_1 + I_2 + I_3.\end{aligned}$$

The situation of I_0 is the same as before: $I_0 \le \frac{2}{1-\alpha}|A|^{\alpha-1}$. For the rest of the cases, we want to use Van der Corput again. Recall that $\phi'(t) = 2t+2A$. For the remaining,

- If $t \in (1/|A|, |A|/2)$, then $2t+2A \in (2/|A|+2A, |A|+2A)$. Observe that $|\phi'(t)| \ge |A|$, since
  - If $A>0$, $2t+2A > 2A+2/|A| > 2A > A = |A|$.
  - If $A<0$, $2t+2A < 2A+|A| = A < 0 \implies |2t+2A| \ge -A = |A|$.

  Therefore, we take $\phi^* = \phi/|A|$ so that $|(\phi^*)'| \ge 1$, and with Van der Corput,

$$|I_1| = \left|\int_{1/|A|}^{|A|/2} e^{i|A|\phi^*}\psi\right| \le C|A|^{-1}\left((|A|/2)^{-\alpha} - t^{-\alpha}\Big|_{1/|A|}^{|A|/2}\right) = C_\alpha|A|^{\alpha-1}.$$

- If $t \in (-|A|/2, -1/|A|)$, then $2t+2A \in (2A-|A|, 2A-2/|A|)$. Also here $|\phi'(t)| \ge |A|$, since
  - If $A>0$, $2t+2A > 2A-|A| = A = |A|$.
  - If $A<0$, $2t+2A < 2A-2/|A| < 2A < 0 \implies |2t+2A| \ge -2A = 2|A| \ge |A|$.

  Therefore, we take again $\phi^* = \phi/|A|$ for the same reason and observing that $|\psi'(t)| = \alpha|t|^{-\alpha-1} = \alpha(-t)^{-\alpha-1}$, we see that

$$\begin{aligned}|I_1^-| &\le C|A|^{-1}\left(|\psi(1/|A|)| + \int_{-|A|/2}^{-1/|A|} |\psi'(t)|\,dt\right) = C|A|^{-1}\left(|A|^{\alpha} - t^{-\alpha}\Big|_{1/|A|}^{|A|/2}\right) \\ &= C|A|^{-1}\left(2|A|^{\alpha} - (|A|/2)^{-\alpha}\right) \le C_\alpha\left(|A|^{\alpha-1} + |A|^{-\alpha-1}\right) \\ &\le C_\alpha\left(|A|^{\alpha-1} + |A|^{-\alpha}\right),\end{aligned}$$

  which satisfies (1.16).

- If $t \in (2|A|, \infty)$, then $2t+2A \in (4|A|+2A, \infty)$, so for positive A, $2t+2A > 6A > |A|$, and for negative A, $2t+2A > -2A = 2|A| = |A|$. Hence, by the trick of ϕ^*, and for $R > 2|A|$,

$$\left|\int_{2|A|}^{R} e^{i|A|\phi^*}\psi\right| \le C|A|^{-1}\left(R^{-\alpha} - t^{-\alpha}\Big|_{2|A|}^{R}\right) = C_\alpha|A|^{-\alpha-1}.$$

  Since this is valid for every $R > 2|A|$, we see that $|I_3| \le C_\alpha|A|^{-\alpha}$ since $|A|>2$.

- If $t \in (-\infty, -2|A|)$, then $2t+2A < -4|A|+2A$. If A is positive, then $2t+2A < -2A < -A$, so $|2t+2A| > |A|$. On the other hand, if it is negative, then $2t+2A < 6A < 0$, so $|2t+2A| > 6|A| > |A|$. Hence, repeating the procedure, for big R,

$$\left|\int_{-R}^{-2|A|} e^{i|A|\phi^*}\psi\right| \le C|A|^{-1}\left((2|A|^{-\alpha} - t^{-\alpha}\Big|_R^{2|A|}\right) = C|A|^{-1}\left(2(2|A|)^{-\alpha} - R^{-\alpha}\right)$$
$$\le C_\alpha |A|^{-1}|A|^{-\alpha},$$

so we see that $|I_3^-| \le C_\alpha |A|^{-\alpha-1} \le C_\alpha |A|^{-\alpha}$ because $|A| > 2$.

The situation is a bit more tricky in the case of I_2 and I_2^-. Indeed,

- If $t \in (|A|/2, 2|A|)$, then $2t+2A \in (|A|+2A, 4|A|+2A)$. If $A > 0$, then $2t+2A > 3A > |A|$, so we can argue the same as before to say that

$$|I_2| \le C|A|^{-1}\left((2|A|)^{-\alpha} - t^{\alpha}\Big|_{|A|/2}^{2|A|}\right) = C_\alpha |A|^{-\alpha-1} \le C_\alpha |A|^{-\alpha}.$$

But if $A < 0$, then $2t+2A \in (A, -2A)$, so we cannot argue the same. But observe that if we change variables $t = r + |A| = r - A$, then

$$\int_{|A|/2}^{2|A|} e^{i(2tA+t^2)} \frac{dt}{|t|^\alpha} = \int_{-|A|/2}^{|A|} e^{i(r^2-A^2)} \frac{dr}{(r+|A|)^\alpha}.$$

Therefore, calling the phase $\phi(r) = r^2 - A^2$, we see that $\phi''(r) = 2 > 1$ so by Corollary A.9, with $\psi(r) = (r+|A|)^{-\alpha}$,

$$\begin{aligned}\left|\int_{-|A|/2}^{|A|} e^{i(r^2-A^2)} \frac{dr}{(r+|A|)^\alpha}\right| &\le C\left(\psi(|A|) + \int_{-|A|/2}^{|A|} |\psi'(x)|\,dx\right) \\ &= C\left((2|A|)^{-\alpha} - (r+|A|)^{-\alpha}\Big|_{-|A|/2}^{|A|}\right) \\ &= C(|A|/2)^{-\alpha} = C_\alpha |A|^{-\alpha},\end{aligned} \tag{1.17}$$

so $|I_2| \le C_\alpha |A|^{-\alpha}$.

- Finally, the case of $t \in (-2|A|, -|A|/2)$ is similar. Indeed, if $A < 0$, then $2t+2A \in (6A, 3A)$, so $2t+2A < 3A < A$ and $|2t+2A| > |A|$. By the trick of ϕ^*, we see that

$$\begin{aligned}|I_2^-| &\le C|A|^{-1}\left((|A|/2)^{-\alpha} - t^{-\alpha}\Big|_{|A|/2}^{2|A|}\right) \\ &= C|A|^{-1}\left(2(|A|/2)^{-\alpha} - (2|A|)^{-\alpha}\right) = C_\alpha |A|^{-\alpha-1},\end{aligned}$$

so in this case $|I_2^-| \le C_\alpha |A|^{-\alpha-1} \le C_\alpha |A|^{-\alpha}$. But if $A > 0$, then $2t+2A \in (-2A, A)$, so we cannot bound ϕ'. But with the change of variables $t = -r - |A|$,

$$\int_{-2|A|}^{-|A|/2} e^{i(2tA+t^2)} \frac{dt}{|t|^\alpha} = \int_{-|A|/2}^{|A|} e^{i(r^2-A^2)} \frac{dr}{(r+|A|)^\alpha},$$

which is exactly the same integral as in (1.17), so $|I_2^-| \le C_\alpha |A|^{-\alpha}$.

Therefore, we conclude that since the bounds of every piece of the integral are either $|A|^{-\alpha}$ or $|A|^{\alpha-1}$ and only with constants depending on α, we deduce that

$$\int_{\mathbb{R}} e^{i(2At+t^2)} \frac{dt}{|t|^{\alpha}} \leq C_{\alpha}\left(|A|^{-\alpha}+|A|^{\alpha-1}\right),$$

which is the result we asked for in (1.16). The lemma is proven. □

1.2 The Negative Result

Carleson's result given in Theorem 1.1 asserts that convergence is obtained whenever $f \in H^s$ for indices $s \geq 1/4$. The natural question arising is whether convergence holds for Sobolev spaces of smaller index. In other words, we wonder if this is a sharp result. In this section we will see that the result is indeed sharp, showing that the index $s = 1/4$ is the best possible. To accomplish this task, after fixing $s < 1/4$, we will build a function in H^s such that the solution given by the operator $e^{it\Delta}$ does not converge to the chosen initial data.

As a first step, it is interesting to find the reasons why the arguments in Section 1.1 do not work this time. Observe that the keystone was the estimate given in Corollary 1.5, an estimate concerning an L^p-norm the maximal function

$$\sup_{t>0}\left|e^{it\Delta}f(x)\right| \tag{1.18}$$

in terms of the $H^{1/4}$-norm of the initial data. We observe that the proof of Proposition 1.3 heavily depends on Lemma 1.2 and on the fact that the case $s = 1/4$ simplifies its bound because it allows us to take $\alpha = 1/2$. But this time we will show that the maximal expression (1.18), when considered as an operator $H^s \to L^p$, is not bounded, thus removing the possibility of repeating the argument. Let us see this.

Let $s < 1/4$, and for $j \in \mathbb{N}$, consider a function $f_j(x)$ defined via its Fourier transform

$$\widehat{f_j}(\xi) = \chi_{[2^j, 2^j+\frac{1}{100}2^{j/2}]}(\xi). \tag{1.19}$$

Let us analyse its H^s norm. For that, we see that

$$\|f_j\|_{H^s}^2 = \int_{2^j}^{2^j+\frac{1}{100}2^{j/2}} (1+|\xi|^2)^s \, d\xi. \tag{1.20}$$

Observe that

$$2^j + \frac{1}{100}2^{j/2} \leq 2^j + 2^{j/2} \leq 2\cdot 2^j,$$

so by the effect of the interval, we see that $2^j \leq \xi \leq 2\cdot 2^j$, from which we deduce $2^{2j} \leq \xi^2 \leq 4\cdot 2^{2j}$, and since $j \geq 1$, we can also say that $2^{2j} \leq 1+\xi^2 \leq 4\cdot 2^{2j}+1 \leq 5\cdot 2^{2j}$. Then,

$$2^{2js} \leq (1+\xi^2)^s \leq 5^s 2^{2js}.$$

This means that from (1.20) we deduce

$$\frac{2^{2js}2^{j/2}}{100} \leq \|f_j\|_{H^s}^2 \leq 5^s \frac{2^{2js}2^{j/2}}{100} \implies \frac{2^{js}2^{j/4}}{10} \leq \|f_j\|_{H^s} \leq 5^{s/2}\frac{2^{js}2^{j/4}}{10}.$$

In other words,

$$\|f_j\|_{H^s} \approx 2^{js}2^{j/4}, \tag{1.21}$$

where the constants of the equivalence are 1/10 and $5^{s/2}/10$, the last depending on s, which does not matter since $s < 1/4$ is fixed.

After estimating the norm of the chosen f_j, we focus on the solution expression, $e^{it\Delta}f_j$, which after a change of variables given by $\xi \to \xi + 2^j$ turns into

$$\left|e^{it\Delta}f_j(x)\right| = \left|\int_0^{\frac{2^{j/2}}{100}} e^{2\pi i(x(\xi+2^j)-2\pi t(\xi+2^j)^2)}\,d\xi\right| = \left|\int_0^{\frac{2^{j/2}}{100}} e^{2\pi i(x\xi-2\pi t\xi^2-4\pi t\xi 2^j)}\,d\xi\right|. \tag{1.22}$$

The terms not depending on ξ have disappeared by the effect of the absolute value. Now, since we are interested in treating $\sup_{t>0}\left|e^{it\Delta}f_j(x)\right|$, we choose a certain value for t so that the phase in the integral (1.22) is really small. We restrict ourselves to $x \in [0,1]$ and we define

$$t_j = 2^{-j-1}(2\pi)^{-1}x. \tag{1.23}$$

Hence, the phase becomes bounded by

$$|x\xi - 2\pi t_j\xi^2 - 4\pi t_j\xi 2^j| \leq \xi(x-x) + 2^{-j-1}x\xi^2 \leq 2^{-j-1}\xi^2 \leq 2^{-j-1}\frac{2^j}{100^2} = \frac{1}{20000}.$$

Now, bounding (1.22) from below with the real part,

$$\left|e^{it_j\Delta}f_j(x)\right| \geq \left|\int_0^{\frac{2^{j/2}}{100}} \cos 2\pi(x\xi - 2\pi t\xi^2 - 4\pi t\xi 2^j)\,d\xi\right|,$$

and the cosine term can be bounded from below by 1/2 because the phase is very close to 0. Hence,

$$\left|e^{it_j\Delta}f_j(x)\right| \geq \frac{1}{2}\frac{2^{j/2}}{100} = C2^{j/2}, \tag{1.24}$$

from where we can assert that

$$\sup_{t>0}\left|e^{it\Delta}f_j(x)\right| \geq \left|e^{it_j\Delta}f_j(x)\right| \geq C2^{j/2}, \quad \forall x \in [0,1].$$

Therefore for any $1 \leq p \leq \infty$ we have

$$\|\sup_{t>0}\left|e^{it\Delta}f_j(x)\right|\|_{L^p(\mathbb{R})} \geq \|\sup_{t>0}\left|e^{it\Delta}f_j(x)\right|\|_{L^p([0,1])} \geq C2^{j/2}. \tag{1.25}$$

The bound (1.25) we have obtained, alongside (1.21), will give the desired result. Indeed, observe that (1.21) can also be written as

$$\|f_j\|_{H^s} \approx 2^{js}2^{j/4} = 2^{j/2}2^{j(s-1/4)} \implies \|f_j\|_{H^s}2^{j(1/4-s)} \approx 2^{j/2},$$

so inserting that in (1.25), we get

$$\|\sup_{t>0}\left|e^{it\Delta}f_j(x)\right|\|_{L^p(\mathbb{R})} \gtrsim 2^{j(1/4-s)}\|f_j\|_{H^s}, \quad \forall j \in \mathbb{N}.$$

Since $s < 1/4$, then $1/4 - s > 0$ and when $j \to \infty$ the hypothetical constant for the bounded operator tends to infinity. In other words, it does not exist, so the operator is not bounded. Also observe that if $s = 1/4$ or more generally $s \geq 1/4$, the dependence of the constant on j disappears. It must be so, since we showed that in the case $p = 4$ the operator is bounded.

We have thus seen that the argument used to prove the positive result will not be correct in case $s < 1/4$. What is more, we want to find a counterexample which shows that convergence fails in this case. For that, since $s < 1/4$, we have $s + 1/4 < 1/2$, so consider α such that $s + 1/4 < \alpha < 1/2$. Let us define

$$f(x) = \sum_{l\geq 2} 2^{-l\alpha} f_l(x), \tag{1.26}$$

where f_l was defined in (1.19). First let us check that $f \in H^s$. Indeed,

$$\|f\|_{H^s} \leq \sum_{l\geq 2} 2^{-l\alpha}\|f_l\|_{H^s},$$

and using the estimate (1.21), we can say that

$$\|f\|_{H^s} \leq C_s \sum_{l\geq 2} 2^{-l\alpha+ls+l/4} = C_s \sum_{l\geq 2} 2^{-l(\alpha-s-1/4)}.$$

Observe that by the choice of α, the exponent is negative since $\alpha - s - 1/4 > 0$, hence the sum converges and $f \in H^s$. It is also clear that the partial sums of the series defining f converge to f in H^s, because

$$\left\| f(x) - \sum_{l\geq 2}^{N} 2^{-l\alpha} f_l(x) \right\|_{H^s} = \left\| \sum_{l\geq N+1}^{\infty} 2^{-l\alpha} f_l(x) \right\|_{H^s} \leq C_s \sum_{l=N+1}^{\infty} 2^{-l(\alpha-s-1/4)},$$

which tends to zero when $N \to \infty$ for being the tail of a finite series. This automatically shows that convergence also holds in L^2 because the L^2-norm is dominated by the H^s-norm.

Since we want to deal with $e^{it\Delta}f$, we need to know if expression (1.26) admits to take the operator inside the sum. We know that $e^{it\Delta}$ is linear, but since we are working with an infinite sum, we should justify this step. We saw in Proposition 0.2 that $e^{it\Delta}$ is an isometry in L^2, so since we know that (1.26) is convergence in L^2, then isometries being continuous, we can assert that $\sum_{l=2}^{N} 2^{-l\alpha} e^{it\Delta} f_l \to e^{it\Delta} f$, showing that

$$e^{it\Delta} f = \sum_{l=2}^{\infty} 2^{-l\alpha} e^{it\Delta} f_l \qquad\qquad \text{in } L^2,$$

and therefore pointwise (because there is a subsequence of the partial sums converging pointwise to f).

Since our objective is to show that $e^{it\Delta}f$ does not converge to f when $t \to 0$, we will again work with certain time values tending to zero, in which the function will be too unstable for convergence. Recall the ones defined in (1.23), and also the bound we gave in (1.24) for coinciding values of $l = j$, $\left|e^{it_j\Delta}f_j(x)\right|$. We want to analyse what happens when $l \neq j$ and to see if we can obtain a similar bound. For that, recall the expression in (1.22).

- If $l < j \Leftrightarrow l - j < 0$, by taking the absolute value inside the integral, we obtain
$$\left|e^{it_j\Delta}f_l(x)\right| \leq \frac{2^{l/2}}{100} = C2^{(l-j)/2}2^{j/2} = C2^{-|j-l|/2}2^{j/2}.$$
- Otherwise if $l > j \Leftrightarrow l - j > 0$, the same procedure does not work, since we obtain
$$\left|e^{it_j\Delta}f_l(x)\right| \leq \left|\frac{2^{l/2}}{100}\right| = C2^{(l-j)/2}2^{j/2} = C2^{|j-l|/2}2^{j/2},$$
which is a terrible bound for $l \gg j$. Therefore, we need another technique. Indeed, observe that the integral to treat is
$$\left|\int_0^{\frac{2^{l/2}}{100}} e^{2\pi i(x\xi - 2\pi t_j\xi^2 - 4\pi t_j\xi 2^l)}\,d\xi\right| = \left|\int_0^{\frac{2^{l/2}}{100}} e^{2\pi i((x-2\pi t_j 2^{l+1})\xi - 2\pi t_j\xi^2)}\,d\xi\right|,$$
which is an oscillatory integral with phase $\phi(\xi) = (x - 2\pi t_j 2^{l+1})\xi - 2\pi t_j\xi^2$. We want to use Van der Corput's results given in Proposition A.6. For that, we need to control the derivative of the phase. Clearly,
$$\phi'(\xi) = x - 2\pi t_j(2^{l+1} + 2\xi) = x - 2^{-j-1}x(2^{l+1} + 2\xi) = x(1 - 2^{-j}(2^j + \xi)),$$
and if we consider $x \in (1/2, 1)$,
$$|\phi'(\xi)| \geq \frac{1}{2}|1 - 2^{-j}(2^l + \xi)|.$$
Observe that since $\xi > 0$, then $2^{-j}(2^l + \xi) > 2^{l-j} \geq 2$ because $l - j \geq 1$. Hence,
$$|\phi'(\xi)| \geq \frac{1}{2}(2^{-j}(2^l + \xi) - 1) \geq \frac{1}{2}(2^{l-j} - 1),$$
and since $2^{l-j} \geq 2$, then $2^{l-j-1} \geq 1$ and
$$|\phi'(\xi)| \geq \frac{1}{2}(2^{l-j} - 2^{l-j-1}) = \frac{1}{2}2^{l-j-1} = \frac{1}{4}2^{l-j} = \frac{2^{|j-l|}}{4}.$$
Thus, define $\phi^*(\xi) = \frac{4}{2^{|j-l|}}\phi(\xi)$ so that $(\phi^*)'(\xi) \geq 1$, and since it is monotonic, we can use Van der Corput's result to assert that
$$\left|\int_0^{\frac{2^{l/2}}{100}} e^{2\pi i\phi(\xi)}\,d\xi\right| = \left|\int_0^{\frac{2^{l/2}}{100}} e^{2\pi i\frac{2^{|j-l|}}{4}\phi^*(\xi)}\,d\xi\right| \leq C\left(2\pi\frac{2^{|j-l|}}{4}\right)^{-1} = K2^{-|j-l|}.$$
The constant from Var der Corput is 4, so $K = 8/\pi$. Also observe that
$$2^{-|j-l|} \leq 2^{-|j-l|}2^{|j-l|/2}2^{j/2} = 2^{-|j-l|/2}2^{j/2},$$
from where we deduce that
$$\left|e^{it_j\Delta}f_l(x)\right| \leq K2^{-|j-l|/2}2^{j/2},$$
which is the same bound as in the case $l < j$.

Therefore, observe that if $l \neq j$, we have obtained

$$\left|e^{it_j\Delta}f_l(x)\right| \leq Q2^{-|j-l|/2}2^{j/2}, \quad \forall x \in (1/2,1), \tag{1.27}$$

where $Q = \max\{1/100, 8/\pi\} = 8/\pi$.

Once we have managed to obtain bounds from below for most of the summands of the solution, it is time to tackle the solution itself as a whole. Since we are looking for divergence, we also want to bound $\left|e^{it_j\Delta}f\right|$ from below with some bound depending of j such that it tends to infinity when t_j tends to zero (and by the choice of t_j, equivalently when j tends to infinity). Write

$$|e^{it_j\Delta}f(x)| = \left|\sum_{l=2}^{\infty} 2^{-l\alpha}e^{it_j\Delta}f_l(x)\right|, \tag{1.28}$$

and by the reverse triangle inequality, since j is fixed now, we write

$$|e^{it_j\Delta}f(x)| \geq 2^{-j\alpha}e^{it_j\Delta}f_j(x) - \sum_{l\neq j} 2^{-l\alpha}e^{it_j\Delta}f_l(x),$$

so that we can use both (1.24) and (1.27). Hence,

$$|e^{it_j\Delta}f(x)| \geq \frac{2^{-j\alpha+j/2}}{200} - Q2^{j/2}\sum_{l\neq j}2^{-l\alpha}2^{-|j-l|/2}. \tag{1.29}$$

We need to treat the sum making a difference between cases $l < j$ and $l > j$.

- When $l < j$, we see that

$$\begin{aligned} 2^{j/2}\sum_{l=2}^{j-1}2^{-l\alpha}2^{-|j-l|/2} &= 2^{j/2}\sum_{l=2}^{j-1}2^{-l\alpha}2^{(-j+l)/2} = \sum_{l=2}^{j-1}2^{l(1/2-\alpha)} \\ &= \frac{2^{j(1/2-\alpha)}-2^{2(1/2-\alpha)}}{2^{(1/2-\alpha)}-1} \leq \frac{2^{j(1/2-\alpha)}}{2^{(1/2-\alpha)}-1}. \end{aligned} \tag{1.30}$$

- When $l > j$,

$$\begin{aligned} 2^{j/2}\sum_{l=j+1}^{\infty}2^{-l\alpha}2^{-|j-l|/2} &= 2^{j/2}\sum_{l=j+1}^{\infty}2^{-l\alpha}2^{(j-l)/2} = 2^{j}\sum_{l=j+1}^{\infty}2^{-l(1/2+\alpha)} \\ &= 2^{j}\frac{2^{-(j+1)(1/2+\alpha)}}{1-2^{-(1/2+\alpha)}} = 2^{j(1/2-\alpha)}\frac{2^{-(\alpha+1/2)}}{1-2^{-(1/2+\alpha)}}. \end{aligned} \tag{1.31}$$

Hence, we can bound (1.29) with

$$\begin{aligned} \left|e^{it_j\Delta}f(x)\right| &\geq 2^{j(1/2-\alpha)}\left[\frac{1}{200} - Q\left(\frac{1}{2^{(1/2-\alpha)}-1} + \frac{2^{-(\alpha+1/2)}}{1-2^{-(1/2+\alpha)}}\right)\right] \\ &= C_\alpha 2^{j(1/2-\alpha)}. \end{aligned}$$

Observe that if $C_\alpha > 0$ holds we are done since $1/2-\alpha > 0$ and $2^{j(1/2-\alpha)} \to \infty$ when $j \to \infty$, showing that the solution tends to infinity. Nevertheless, the argument is not finished. Indeed, since

α could be very close to 1/2, it could happen that the constant C_α were negative because the negative term is bigger than 1/200. One way to fix this problem is to remove terms from the initial definition for f. In (1.26), we filter summands by choosing $M \in \mathbb{N}$, $M \gg 1$ and letting

$$f_M(x) = \sum_{l\geq 2} 2^{-Ml\alpha} f_{Ml}(x).$$

Observe that we are doing nothing but considering only one out of M former coefficients. Consider also times t_{Mj} alone. Going back to (1.28), we see that we can follow the same process to write

$$\begin{aligned}|e^{it_{Mj}\Delta} f_M(x)| &\geq 2^{-Mj\alpha} e^{it_{Mj}\Delta} f_{Mj}(x) - \sum_{l\neq j} 2^{-Ml\alpha} e^{it_{Mj}\Delta} f_{Ml}(x) \\ &\geq \frac{2^{-Mj\alpha+Mj/2}}{200} - Q2^{Mj/2} \sum_{l\neq j} 2^{-Ml\alpha} 2^{-M|j-l|/2}.\end{aligned}$$

If we review the calculations in (1.30) and (1.31), one can very easily observe that the effect of the parameter M is translated to the constant C_α which will now be $C_{\alpha,M}$. Indeed, the bounds we get are

$$\begin{aligned}\left|e^{it_j\Delta} f_M(x)\right| &\geq 2^{Mj(1/2-\alpha)} \left[\frac{1}{200} - Q\left(\frac{1}{2^{M(1/2-\alpha)}-1} + \frac{2^{-M(\alpha+1/2)}}{1-2^{-M(1/2+\alpha)}}\right)\right] \\ &\geq 2^{Mj(1/2-\alpha)} \left[\frac{1}{200} - Q\left(\frac{1}{2^{M(1/2-\alpha)}-1} + \frac{2}{2^{M(\alpha+1/2)}}\right)\right] \\ &= C_{\alpha,M} 2^{j(1/2-\alpha)},\end{aligned}$$

where in the last inequality we have chosen M big enough so that $1-2^{-M(1/2+\alpha)} \geq 1/2$. Now, it is clear that

$$\frac{1}{2^{M(1/2-\alpha)}-1}, \frac{2}{2^{M(\alpha+1/2)}} \to 0, \quad \text{when } M \to \infty,$$

so we can choose M big enough to make the sum of those terms smaller than 1/200 and thus $C_{\alpha,M} > 0$. Finally letting $j \to \infty$, we eventually deduce that

$$\lim_{j\to\infty} \left|e^{it_j\Delta} f_M(x)\right| = \infty,$$

showing that

$$\lim_{t\to 0} \left|e^{it\Delta} f_M(x)\right| = \infty$$

and that the solution does not converge to the initial data, even if we saw that $f_M \in H^s$.

2

Analysis in Several Dimensions

Through calculations in Chapter 1 we have been able to characterise convergence in terms of Sobolev spaces H^s in one dimension, this is to say, in $\mathbb{R}$. More precisely, we have seen that the border exponent is $s = 1/4$, with convergence holding if and only if $s \geq 1/4$. We want to see if we can obtain similar results in higher dimensions.

2.1 A Positive Result in $\mathbb{R}^n$

In this section, we will analyse a positive result which gives a sufficient condition for convergence in every dimension. We can write it in the following way.

Theorem 2.1. *Let $s > 1/2$. Then, if $f \in H^s(\mathbb{R}^n)$,*

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x)$$

almost everywhere.

As we see, this result is not as strong as Theorem 1.1 in the sense that the Sobolev exponent is worse. On the other hand, it is a powerful theorem since it ensures convergence in every dimension. It is a fact that in higher dimensions finding the right exponent is tougher and it has not been solved so far as we will later see.

Theorem 2.1 is proved by a similar procedure as the one used to prove Theorem 1.1. Indeed, we will prove an estimate for solutions concerning the maximal function in time of $e^{it\Delta} f$ by means of the Sobolev norm of the data f. We present it in the following proposition.

Proposition 2.2. *Let $s > 1/2$ and $f \in \mathscr{S}(\mathbb{R}^n)$. Then, there exists a constant $C_s > 0$ such that*

$$\left\| \sup_{t>0} \left| e^{it\Delta} f \right| \right\|_{L^2(B_R)} \leq C_s R^{1/2} \| f \|_{H^s(\mathbb{R}^n)}$$

for every $R > 0$, where B_R denotes the ball in $\mathbb{R}^n$ centred at the origin and of radius R.

Proof. In the way we did in the proof of Proposition 1.3, for every $x \in \mathbb{R}^n$ we can find a time value $t(x)$ such that

$$\sup_{t>0} \left|e^{it\Delta} f\right| \geq \left|e^{it(x)\Delta} f\right| \geq \frac{1}{2} \sup_{t>0} \left|e^{it\Delta} f\right|.$$

Therefore, $\sup_{t>0} \left|e^{it\Delta} f\right| \leq 2|e^{it(x)\Delta} f|$ and it is enough to obtain the estimate for $|e^{it(x)\Delta} f(x)|$, thus not needing to work with the supremum in time. By duality, since the dual space of L^2 is L^2 itself, we can find a function $w \in L^2(B_R)$ (which since we only mind the situation in the ball, can be chosen w to be supported in B_R) so that $\|w\|_{L^2(B_R)} = 1$ and

$$\|e^{it(\cdot)\Delta} f(\cdot)\|_{L^2(B_R)} = \int_{B_R} e^{it(x)\Delta} f(x) w(x)\, dx.$$

Because of that, we are looking for a bound like

$$\int_{B_R} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t(x)|\xi|^2)} w(x)\, d\xi\, dx \leq C_s R^{1/2} \|f\|_{H^s} \|w\|_{L^2} \tag{2.1}$$

for functions $w \in L^2(B_R)$.

Observe that the integral in (2.1) admits a change of the order of integration. Indeed, since f is a Schwartz function, so is its Fourier transform and hence $\widehat{f} \in L^1$. Also, $L^2(B_R) \subset L^1(B_R)$, so w is integrable. Therefore, Fubini's theorem can be applied and the integral is

$$\begin{aligned}
&\int_{B_R} \widehat{f}(\xi) \int_{\mathbb{R}^n} e^{2\pi i (x\cdot\xi - 2\pi t(x)|\xi|^2)} w(x)\, dx\, d\xi \\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi)(1+|\xi|^2)^{s/2} \left(\int_{B_R} \frac{e^{2\pi i (x\cdot\xi - 2\pi t(x)|\xi|^2)}}{(1+|\xi|^2)^{s/2}} w(x)\, dx \right) d\xi \\
&\leq \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{\left| \int_{B_R} e^{2\pi i (x\cdot\xi - 2\pi t(x)|\xi|^2)} w(x)\, dx \right|^2}{(1+|\xi|^2)^s}\, d\xi \right)^{1/2},
\end{aligned} \tag{2.2}$$

where the last inequality is nothing but the consequence of applying Hölder's inequality in the outer integral. Now observe that the first term in (2.2) is precisely the H^s norm of the initial data f, since

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi.$$

Hence, we need to work with the second term. For its management, we need to define the following sets. For every $j = 1, \ldots, n$, let

$$U_j = \left\{ \xi \in \mathbb{R}^n \mid |\xi_j| \geq \frac{|\xi|}{\sqrt{n}} \right\}. \tag{2.3}$$

Observe that if $\xi \in \mathbb{R}^n$ were a point for which $|\xi_j| < |\xi|/\sqrt{n}$ for all $j = 1, \ldots, n$, then

$$|\xi|^2 < \sum_{j=1}^{n} \frac{|\xi|^2}{n} = |\xi|^2,$$

which is obviously false, so there exists some j such that $\xi \in U_j$. This shows that

$$\bigcup_{j=1}^{n} U_j = \mathbb{R}^n.$$

What we do next is to split the second integral in (2.2) into these sets U_j. Define

$$I_j = \int_{U_j} \frac{\left|\int_{B_R} e^{2\pi i(x\cdot\xi - 2\pi t(x)|\xi|^2)} w(x)\,dx\right|^2}{(1+|\xi|^2)^s}\,d\xi$$

so that, since U_j are not disjoint, we can bound the integral as

$$\int_{\mathbb{R}^n} \frac{\left|\int_{B_R} e^{2\pi i(x\cdot\xi - 2\pi t(x)|\xi|^2)} w(x)\,dx\right|^2}{(1+|\xi|^2)^s}\,d\xi \le \sum_{j=1}^{n} I_j. \tag{2.4}$$

Therefore, we will work to bound each of these partial integrals I_j. Let us work with I_1 for simplicity. Observe that since I_1 is an $\mathbb{R}^n$ integral of a positive function, we can split it in variables. Call $\xi = (\xi_1, \xi_2, \cdots, \xi_n) = (\xi_1, \overline{\xi})$, where $\overline{\xi} \in \mathbb{R}^{n-1}$. Hence, after splitting,

$$I_1 \le \int_{\overline{U_1}} \int_{U_1(\xi_1)} \frac{\left|\int_{B_R} e^{2\pi i(x_1\xi_1 + \overline{x}\cdot\overline{\xi} - 2\pi t(x)|\xi|^2)} w(x)\,dx\right|^2}{(1+|\xi|^2)^s}\,d\xi_1\,d\overline{\xi},$$

where $\overline{\xi} \in \overline{U_1} \subset \mathbb{R}^{n-1}$. We want to change variables to make $|\xi|^2 = r$. But observe that it cannot be done directly. In fact, we are only going to change variables in the first variable, so to make $|\xi|^2$ become the new variable, we need

$$\xi_1 = \sqrt{r - \overline{\xi}^2}, \qquad d\xi = \frac{1}{2}(r - \overline{\xi}^2)^{-1/2}\,dr = \frac{dr}{2\xi_1}. \tag{2.5}$$

Then,

$$I_1 \le \int_{\overline{U_1}} \int_{\tilde{U}_1(\xi_1)} \frac{\left|\int_{B_R} e^{2\pi i(x_1\xi_1 + \overline{x}\cdot\overline{\xi} - 2\pi t(x)r)} w(x)\,dx\right|^2}{(1+r)^s}\,\frac{dr}{2|\xi_1|}\,d\overline{\xi},$$

and since in U_1 we have $|\xi_1| \ge |\xi|/\sqrt{n}$, we can bound

$$I_1 \le \int_{\overline{U_1}} \int_{\tilde{U}_1(\xi_1)} \frac{\left|\int_{B_R} e^{2\pi i(x_1\xi_1 + \overline{x}\cdot\overline{\xi} - 2\pi t(x)r)} w(x)\,dx\right|^2}{(1+r)^s}\,\frac{\sqrt{n}}{2|\xi|}\,dr\,d\overline{\xi}.$$

But looking at (2.5) we see that indeed $r = |\xi|^2$, so

$$I_1 \le \int_{\overline{U_1}} \int_{\tilde{U}_1(\xi_1)} \frac{\left|\int_{B_R} e^{2\pi i(x_1\xi_1 + \overline{x}\cdot\overline{\xi} - 2\pi t(x)r)} w(x)\,dx\right|^2}{(1+r)^s}\,\frac{\sqrt{n}}{2\sqrt{r}}\,dr\,d\overline{\xi}.$$

Here, we choose to let the term $\xi_1 = \xi_1(r, \overline{\xi})$ in the exponential, since we are to see that it will play no role. Now, since we have an integral of positive functions, we can change the order and integrate in $(0,\infty)$ (because $r = |\xi|^2 \ge 0$) to write

$$I_1 \le \frac{\sqrt{n}}{2} \int_0^\infty \frac{1}{\sqrt{r}(1+r)^s} \int_{\overline{U_1}} \left|\int_{B_R} e^{2\pi i(x_1\xi_1 + \overline{x}\cdot\overline{\xi} - 2\pi t(x)r)} w(x)\,dx\right|^2 d\overline{\xi}\,dr.$$

Let us focus now on the integral on the ball. Since w is integrable, we can by Fubini split the integral to write

$$
\begin{aligned}
&\left|\int_{B_R} e^{2\pi i(x_1\xi_1+\overline{x}\cdot\overline{\xi}-2\pi t(x)r)} w(x)\,dx\right|^2 \\
&= \left|\int_{-R}^{R} e^{2\pi i x_1\xi_1}\left(\int_{\overline{B_R}} e^{2\pi i(\overline{x}\cdot\overline{\xi}-2\pi t(x)r)} w(x)\,d\overline{x}\right) dx_1\right|^2 \\
&= \int_{-R}^{R}\left|e^{2\pi i x_1\xi_1}\right|^2 dx_1 \int_{-R}^{R}\left|\int_{\overline{B_R}} e^{2\pi i(\overline{x}\cdot\overline{\xi}-2\pi t(x)r)} w(x)\,d\overline{x}\right|^2 dx_1,
\end{aligned}
\tag{2.6}
$$

where we have used Cauchy-Schwarz' inequality in the outer integral. Observe that the first term in (2.6) is nothing but $2R$. Hence, taking the expression for I_1 again and extending the integral from both $\overline{B_R}$ and $\overline{U_1}$ to $\mathbb{R}^{n-1}$ we write

$$
I_1 \leq \sqrt{n}R\int_0^\infty \frac{1}{\sqrt{r}(1+r)^s}\int_{-R}^{R}\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}^{n-1}} e^{2\pi i(\overline{x}\cdot\overline{\xi}-2\pi t(x)r)} w(x)\,d\overline{x}\right|^2 d\overline{\xi}\,dx_1\,dr \tag{2.7}
$$

after several Fubini's changes of order. Observe that $\overline{U_1}$ can be trivially extended to the whole space. On the other hand, in principle we cannot extend the integral from $\overline{B_R}$ since we are integrating a complex function, but since w is supported in the ball, it will be null outside, so we can do it.

Plancherel's identity tells us that the Fourier transform is an isometry in L^2. We use this fact to simplify the two innermost integrals, because writing

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}^{n-1}} e^{2\pi i(\overline{x}\cdot\overline{\xi}-2\pi t(x)r)} w(x)\,d\overline{x}\right|^2 d\overline{\xi} \\
&= \int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}^{n-1}} w(x)e^{-4\pi^2 i t(x)r} e^{2\pi i\overline{x}\cdot\overline{\xi}}\,d\overline{x}\right|^2 d\overline{\xi} \\
&= \int_{\mathbb{R}^{n-1}}\left|\mathscr{F}_{\overline{x}}\left(w(x)e^{-4\pi^2 i t(x)r}\right)(\overline{\xi})\right|^2 d\overline{\xi},
\end{aligned}
$$

we obtain the L^2 norm of a Fourier transform. Then, noting that we are not integrating over x_1, by Plancherel this is the same as

$$
\int_{\mathbb{R}^{n-1}}\left|w(x)e^{-4\pi^2 i t(x)r}\right|^2 d\overline{x} = \int_{\mathbb{R}^{n-1}} |w(x)|^2\,d\overline{x}.
$$

Coming back to (2.7), we see that

$$
\begin{aligned}
I_1 &\leq \sqrt{n}R\int_0^\infty \frac{1}{\sqrt{r}(1+r)^s}\int_{-R}^{R}\int_{\mathbb{R}^{n-1}} |w(x)|^2\,d\overline{x}\,dx_1\,dr \\
&= \sqrt{n}R\,\|w\|_{L^2}^2\int_0^\infty \frac{1}{\sqrt{r}(1+r)^s}\,dr.
\end{aligned}
$$

The remaining integral can be easily managed as follows:

$$
\begin{aligned}
\int_0^\infty \frac{1}{\sqrt{r}(1+r)^s}\,dr &= \int_0^1 \frac{1}{\sqrt{r}(1+r)^s}\,dr + \int_1^\infty \frac{1}{\sqrt{r}(1+r)^s}\,dr \\
&\leq \int_0^1 \frac{1}{\sqrt{r}}\,dr + \int_1^\infty \frac{1}{\sqrt{r}r^s}\,dr \\
&= 2 + \int_1^\infty \frac{1}{r^{s+1/2}}\,dr.
\end{aligned}
$$

The last integral is finite if and only if $s+1/2>1$, which is the same as saying that $s>1/2$. Therefore, if we call this quantity K_s, we see that

$$I_1 \le K_s \sqrt{n} R \|w\|_{L^2}^2.$$

Observe that the bound for the remaining I_j is done exactly the same way if we separate x_j instead of x_1. Then, from (2.2) and (2.4) we get the bound

$$\|f\|_{H^s}(K_s n \sqrt{n} R \|w\|_{L^2}^2)^{1/2} = K_{s,n} R^{1/2} \|f\|_{H^s} \|w\|_{L^2},$$

as we asked in (2.1), so we are done.

□

As we did in Chapter 1, the estimate obtained in Proposition 2.2 can be extended to general functions in H^s. This step is needed because we are looking for results in H^s.

Corollary 2.3. *Let $s>1/2$ and $f \in H^s(\mathbb{R}^n)$. Then, there exists a constant $C_s>0$ such that*

$$\| \sup_{t>0} |e^{it\Delta} f| \|_{L^2(B_R)} \le C_s R^{1/2} \|f\|_{H^s(\mathbb{R}^n)}$$

for every $R>0$.

Again, the fact that Schwartz functions are dense in the Sobolev spaces will give the way to prove this result. The approximation argument is exactly the same as which we used to prove Corollary 1.7, the only differences being that now we have $L^2(B_R)$ instead of $L^4(\mathbb{R})$ and that we substitute $H^{1/4}(\mathbb{R})$ for a more general $H^s(\mathbb{R}^n)$, changes that do not affect the argument. For that, we will omit the proof here and we refer the reader to the proof of Corollary 1.7.

Finally, we are ready to use the estimate in Corollary 2.3 to complete the proof of Theorem 2.1. The argument we will perform is very similar to that we used to prove Theorem 1.1, but in this case we have estimates in balls B_R which depend on the radius $R>0$ instead of the universal estimate we had there. That obliges us to be more careful, but as we will see it will not trouble much.

Proof of Theorem 2.1. Proposition 2.2 combined with the same procedure used to prove Theorem 1.1 in Chapter 1 implies that

$$m\left(\left\{x \in B_R \mid \limsup_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| \neq 0\right\}\right) = 0, \quad \forall R>0.$$

Hence we conclude by saying that

$$m\left(\left\{x \in \mathbb{R}^n \mid \limsup_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| \neq 0\right\}\right) \le \sum_{k=1}^{\infty} m\left(\left\{x \in B_k \mid \limsup_{t\to 0} \left|e^{it\Delta} f(x) - f(x)\right| \neq 0\right\}\right) = 0.$$

□

2.2 A Necessary Condition

We have seen in Section 2.1 that a sufficient condition for convergence in $\mathbb{R}^n$ is $s > 1/2$. We wonder if, in the same way we saw that in $\mathbb{R}$, this condition is also necessary. For what it is known so far, this is way too much to say, but nevertheless we will be able to give a necessary condition, which will show that Theorem 2.1 is quite strict taking into account that it is a result independent of the dimension.

As suggested, the result we are to analyse in the following pages gives a necessary condition for convergence in $\mathbb{R}^n$ when $n \geq 3$. It is a result due to R. Lucà and K. M. Rogers which was obtained in [8]. It can be stated as follows.

Theorem 2.4. *Let $n \geq 3$ and consider that*

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$$

almost everywhere for every function $f \in H^s(\mathbb{R}^n)$. Then, $s \geq \frac{1}{2} - \frac{1}{n+2}$.

Observe that if this theorem is read the other way around, it asserts that for values $s < \frac{1}{2} - \frac{1}{n+2}$ convergence does not occur in general, so there exist counterexamples in which the solution does not converge to the initial data. Also observe that the border value obtained in this theorem satisfies

$$\lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} = \frac{1}{2}.$$

This fact shows that the result in Theorem 2.1 is almost strict (the case $s = 1/2$ being the prospective improvement) in the sense that the border $s = 1/2$ cannot be improved if we seek a universal exponent which is valid for all dimensions.

Let us explain the argument for proving Theorem 2.4. Indeed, recall that when we have sought positive results, we have first obtained maximal estimates in terms of the H^s norm of the initial data. What we will see is that if we have an estimate of this kind, then the condition for s in Theorem 2.4 must hold. Of course, in principle this would not be enough, since there might be methods to prove convergence other than the maximal estimate method. Nevertheless, it can be proven that the almost everywhere convergence implies a maximal estimate in L^2. This is a consequence of applying the Stein-Nikishin maximal principle, which generates a weak L^2 estimate, and the interpolation technique in [1, Proof of Lemma C.1] which makes it become into a strong estimate. Hence, it will be enough to prove

Theorem 2.5. *Let $n \geq 3$ and assume there exists a constant $C_s > 0$ such that*

$$\Big\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \Big\|_{L^2(B(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}$$

for every $f \in \mathscr{S}(\mathbb{R}^n)$. Then, $s \geq \frac{1}{2} - \frac{1}{n+2}$.

Before we start proving Theorem 2.5, we will present a lemma regarding the δ-density of a particular set which, as we will see, will be critical in the proof.

Definition 2.6. Let E and F be sets in $\mathbb{R}^n$ and $\delta > 0$. We say that E is δ**-dense** in F if for every $x \in F$ there exists a point $y \in E$ such that $|x - y| < \delta$.

Lemma 2.7. *Let $n \geq 3$ and $0 < \sigma < \frac{1}{n+2}$. For some direction $\theta \in \mathbb{S}^{n-1}$ we define the set*

$$E_\theta = \bigcup_{t \in R^{2\sigma-1}\mathbb{Z} \cap (0,1)} \{x \in R^{\sigma-1}\mathbb{Z}^n : |x| < 2\} + t\theta.$$

Then, for any $\epsilon > 0$, there exists a value $\theta \in \mathbb{S}^{n-1}$ such that E_θ is ϵR^{-1}-dense in the ball $B(0,1/2)$ for every $R \gg 1$.

Proof. What we do first is to rescale the problem. Indeed, if we dilate $x \to R^{1-\sigma}x$ in space, we will see that the initial objective, which can be written as

$$\forall |x| < \frac{1}{2}, \quad \exists y \in E_\theta \quad : \quad |x - y| < \epsilon R^{-1},$$

and since $|x - y| < \epsilon R^{-1}$ is equivalent to $R^{1-\sigma}|x - y| < \epsilon R^{-\sigma}$, it is the same as asking that

$$\forall |x| < \frac{R^{1-\sigma}}{2}, \quad \exists y \in R^{1-\sigma}E_\theta \quad : \quad |x - y| < \epsilon R^{-\sigma},$$

this is to say, that $R^{1-\sigma}E_\theta$ is $\epsilon R^{-\sigma}$-dense in $B(0, R^{1-\sigma}/2)$. And it is also easy to see that

$$R^{1-\sigma}E_\theta = E_\theta = \bigcup_{t \in R^{\sigma}\mathbb{Z} \cap (0,R^{1-\sigma})} \{x \in \mathbb{Z}^n : |x| < 2R^{1-\sigma}\} + t\theta. \tag{2.8}$$

Hence, we are looking for the $\epsilon R^{-\sigma}$-density of (2.8) in $B(0, R^{1-\sigma}/2)$, or in other words, for $x \in B(0, R^{1-\sigma}/2)$, we need to find $\theta \in \mathbb{S}^{n-1}$, $y_x \in \mathbb{Z}^n \cap B(0, 2R^{1-\sigma})$ and $t_x \in R^{\sigma}\mathbb{Z} \cap (0, R^{1-\sigma})$ such that

$$|x - y_x - t_x\theta| < \epsilon R^{-\sigma}.$$

Let us work in the quotient space $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. Recall that in this space, we have an equality $[x] = [y]$ if $x - y \in \mathbb{Z}^n$. Hence, we are partitioning $\mathbb{R}^n$ into cubes of length 1 and with corners at $\mathbb{Z}^n$, so that every cube is the same. Therefore, we are basically working on a cube $\mathbb{T}^n \approx [0,1]^n$. Let us suppose that for a class $[x] \in \mathbb{T}^n$ we can find $t_x \in R^{\sigma}\mathbb{Z} \cap (0, R^{1-\sigma})$ such that

$$|[x] - [t_x\theta]| < \epsilon R^{-\sigma}. \tag{2.9}$$

In that case, if we take any $x \in B(0, R^{1-\sigma}/2)$, then by (2.9) we have a value $t_x \in R^{\sigma}\mathbb{Z} \cap (0, R^{1-\sigma})$. Consider $z_x \in \mathbb{R}^n$ such that $[z_x] = [t_x]$ and such that z_x lies in the same original cube as x. Then, considering $y_x = z_x - t_x \in \mathbb{Z}^n$, we see that $y_x \in \mathbb{Z}^n$ and

$$|x - y_x - t_x| = |x - z_x| = |[x] - [t_x]| < \epsilon R^{-\sigma}.$$

Moreover, the reverse triangle inequality asserts that

$$|y_x| < |x| + |z_x| + \epsilon R^{-\sigma} < \frac{3}{2} R^{1-\sigma} + \epsilon R^{-\sigma} < \left(\frac{3}{2} + \epsilon\right) R^{1-\sigma},$$

because $R > 1$. Hence, a choice of $\epsilon < 1/2$ gives $y_x \in B(0, 2R^{1-\sigma})$ as desired. Hence, let us prove (2.9).

Consider a smooth function $\phi : \mathbb{T}^n \to [0, (2/\epsilon)^n)$ with support in a small ball $B(0, \epsilon/2)$. Since the maximum integral value for ϕ is $c_n(\epsilon/2)^n(2/\epsilon)^2 = c_n > 1$ (where c_n is the measure of the n-dimensional unit ball), we can make it have $\int_{\mathbb{T}^n} \phi = \int_{B(0,(\epsilon/2))} \phi = 1$. Now call the dilation

$$\phi_R(x) = \phi(R^\sigma x).$$

Observe that the support of ϕ_R is $B(0, \epsilon R^{-\sigma}/2)$. Also an easy change of variables shows that

$$\int_{B(0,\epsilon R^{-\sigma}/2)} \phi_R(x)\, dx = R^{-n\sigma} \int_{B(0,\epsilon/2)} \phi(x)\, dx = R^{-n\sigma}. \tag{2.10}$$

If we were able to find a direction $\theta \in \mathbb{S}^{n-1}$ such that for any $x \in \mathbb{T}^n$ we could find $t_x \in (R^\sigma \mathbb{Z} + [-\epsilon R^{-\sigma}/2, \epsilon R^{-\sigma}/2]) \cap (0, R^{1-\sigma})$ satisfying $\phi_R(x - t_x\theta) > 0$, we would obtain (2.9). Indeed, the fact that the value of the function is positive implies that the point is in the support, so $|x - t_x\theta| < \epsilon R^{-\sigma}/2$. And since we have allowed a margin for t_x, then considering the closest $t'_x \in R^\sigma \mathbb{Z} \cap (0, R^{1-\sigma})$, by the triangle inequality we have

$$|x - t'_x\theta| < |x - t_x\theta| + |t_x\theta - t'_x\theta| < \frac{\epsilon}{2} R^{-\sigma} + \frac{\epsilon}{2} R^{-\sigma} = \epsilon R^{-\sigma}.$$

Let ψ be another smooth function $\psi : \mathbb{R} \to [0, 2/\epsilon)$ with support in $(-\epsilon/2, \epsilon/2)$. Observe that the maximum integral value is $2\epsilon/2 \cdot 2\epsilon = 2$, so we can choose it so that $\int \psi = 1$. Using this function, we define

$$\eta_R(t) = R^{3\sigma-1} \sum_{j=1}^{\lfloor R^{1-2\sigma} \rfloor} \psi(R^\sigma(t - R^\sigma j)), \tag{2.11}$$

which has clearly support contained $R^\sigma \mathbb{Z} + (-\epsilon R^{-\sigma}/2, \epsilon R^{-\sigma}/2)$. It is also chosen that way for it to have integral

$$\begin{aligned} \int_{\mathbb{R}} \eta_R(t)\, dt &= R^{3\sigma-1} \sum_{j=1}^{\lfloor R^{1-2\sigma} \rfloor} \int_{\mathbb{R}} \psi(R^\sigma(t - R^\sigma j))\, dt \\ &= R^{3\sigma-1} \sum_{j=1}^{\lfloor R^{1-2\sigma} \rfloor} R^{-\sigma} \int_{\mathbb{R}} \psi(t)\, dt = R^{2\sigma-1} \lfloor R^{1-2\sigma} \rfloor \approx 1. \end{aligned} \tag{2.12}$$

Again, if we are able to find a direction $\theta \in \mathbb{S}^{n-1}$ so that

$$\int \phi_R(x - t\theta)\eta_R(t)\, dt > 0, \qquad \forall x \in \mathbb{T}^n, \tag{2.13}$$

then since we are integrating in $\operatorname{supp} \eta_R$ and since both ϕ_R and η_R are positive functions, there must exist $t \in \operatorname{supp} \eta_R$ so that $\phi_R(x - t\theta) > 0$. Also $\eta_R(t) > 0$, so by the support conditions,

$$t \in R^\sigma j + (-\epsilon R^{-\sigma}/2, \epsilon R^{-\sigma}/2), \qquad j = 1, \dots \lfloor R^{1-2\sigma} \rfloor,$$

and because of $\epsilon R^{-\sigma} \ll 1$, we deduce $t > 0$. Also, since $j < R^{1-2\sigma}$, $t < R^{1-\sigma}$, so the properties asked for t are satisfied. Hence it is enough to prove (2.13).

Use the Fourier series expression of ϕ_R to write that of $\phi(x-t\theta)$. This way, we see that

$$\phi_R(x-t\theta) = \sum_{k\in\mathbb{Z}^n} \widehat{\phi}_R(k)e^{2\pi i k\cdot(x-t\theta)} = \widehat{\phi}_R(0) + \Gamma(x,t,\theta),$$

where Γ is the sum pf every values excepting $k=0$. Recalling (2.10) and (2.12), we can write

$$\begin{aligned}\int_{\mathbb{R}} \phi_R(x-t\theta)\eta_R(t)\,dt &= \int_{\mathbb{R}} \widehat{\phi}_R(0)\eta_R(t)\,dt + \int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt \\ &\approx R^{-n\sigma} + \int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt.\end{aligned} \tag{2.14}$$

because $\widehat{\phi}_R(0) = \int \phi$. Then, since we want (2.14) to be positive for big enough $R \gg 1$, it will be enough to ask for

$$\left|\int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt\right| \lesssim R^{-\gamma}, \qquad \forall x \in \mathbb{T}^n, \tag{2.15}$$

for some $\gamma > n\sigma$. Indeed, if (2.15) holds, then the reverse triangle inequality shows that

$$\begin{aligned}R^{-n\sigma} + \int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt &\geq R^{-n\sigma} - \left|\int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt\right| \\ &\geq R^{-n\sigma} - CR^{-\gamma}\end{aligned}$$

for some constant $C > 0$. And of course, that will be positive if

$$R^{-n\sigma} > CR^{-\gamma} \Leftrightarrow R^{\gamma-n\sigma} > C.$$

Since $\gamma - n\sigma > 0$, we see that $\lim_{R\to\infty} R^{\gamma-n\sigma} = \infty$, so the the positivity we ask for will hold for big enough $R \gg 1$.

Hence, consider the integral and write

$$\begin{aligned}\left|\int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt\right| &= \left|\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^n-\{0\}} \widehat{\phi}_R(k)e^{2\pi i k\cdot(x-t\theta)}\eta_R(t)\,dt\right| \\ &= \left|\sum_{k\in\mathbb{Z}^n-\{0\}} \widehat{\phi}_R(k)\int_{\mathbb{R}} e^{2\pi i k\cdot(x-t\theta)}\eta_R(t)\,dt\right| \\ &\leq \sum_{k\in\mathbb{Z}^n-\{0\}} \left|\widehat{\phi}_R(k)\right| \left|\int_{\mathbb{R}} e^{2\pi i k\cdot(x-t\theta)}\eta_R(t)\,dt\right|,\end{aligned} \tag{2.16}$$

where we have made use of Fubini's theorem to write the second equality. To justify it, and by (2.12), we need to check that

$$\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^n-\{0\}} \left|\widehat{\phi}_R(k)\right|\eta_R(t)\,dt = \sum_{k\in\mathbb{Z}^n-\{0\}} \left|\widehat{\phi}_R(k)\right| < \infty.$$

Observe that

$$\begin{aligned}\widehat{\phi}_R(k) = \int_{B(0,\epsilon R^{-\sigma}/2)} \phi(R^\sigma x)e^{-2\pi i k\cdot x}\,dx &= R^{-n\sigma}\int_{B(0,\epsilon/2)} \phi(x)e^{-2\pi i k\cdot R^{-\sigma}x}\,dx \\ &= R^{-n\sigma}\widehat{\phi}(R^{-\sigma}k)\end{aligned}$$

Since ϕ is smooth and compactly supported, it is a Schwartz function, so there exists a constant $C > 0$ such that $(1+|x|)^{n+1}|\widehat{\phi}(x)| \le C$, and hence

$$|\widehat{\phi}_R(k)| \le \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}}, \tag{2.17}$$

so we need to solve

$$\sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}}. \tag{2.18}$$

Observe that it is a Riemann sum, so we can bound it by means of an integral,

$$\sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}} \lesssim \int_{\mathbb{R}^n} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}}\,dk = \int_{\mathbb{R}^n} \frac{dk}{(1+|k|)^{n+1}},$$

and since we have a radial function, the polar coordinates lead us to

$$\int_0^\infty \frac{r^{n-1}}{(1+r)^{n+1}}\,dr < \int_0^1 r^{n-1}\,dr + \int_1^\infty \frac{r^{n-1}}{r^{n+1}}\,dr < \infty.$$

Hence, the sum is finite (and we can bound it with a constant independent of σ) and Fubini's theorem can be applied. So going back to (2.16), by (2.17) we see that

$$\left|\int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt\right| \le \sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}} \left|\widehat{\eta_R}(k\cdot\theta)\right|, \tag{2.19}$$

since the $k\cdot x$ term of the exponential can be eliminated with the absolute value. Observe that this bound we have obtained is independent of x. Since we want to obtain (2.15) for some $\theta \in \mathbb{S}^{n-1}$, we will consider the integral in $\mathbb{S}^{n-1}$ to see that the mean value of the function can be bounded by what interests us, $R^{-n\sigma}$. This way, there will be a point $\theta \in \mathbb{S}^{n-1}$ for which the function is smaller than a constant times $R^{-n\sigma}$ (otherwise the mean would be greater than $R^{-n\sigma}$). Hence, let us analyse

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} &\sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}} \left|\widehat{\eta_R}(k\cdot\theta)\right| d\theta \\ &= \sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}} \int_{\mathbb{S}^{n-1}} \left|\widehat{\eta_R}(k\cdot\theta)\right| d\theta,\end{aligned} \tag{2.20}$$

where the equality is a consequence of Fubini's theorem for positive functions. Let us work out the transform of η_R. Indeed, from the definition at (2.11),

$$\widehat{\eta_R}(t) = R^{3\sigma-1} \sum_{j=1}^{\lfloor R^{1-2\sigma}\rfloor} \mathscr{F}\left(\psi(R^\sigma(t-R^\sigma j))\right)(t),$$

and the usual properties for translations and dilations of the Fourier transform (by changes of variables) lead to

$$\mathscr{F}\left(\psi(R^\sigma(t-R^\sigma j))\right)(t) = R^{-\sigma} e^{-2\pi i t R^\sigma j}\,\widehat{\psi}(R^{-\sigma}t).$$

Thus,

$$\widehat{\eta_R}(t) = R^{2\sigma-1}\widehat{\psi}(R^{-\sigma}t)\sum_{j=1}^{\lfloor R^{1-2\sigma}\rfloor} e^{-2\pi i t R^{\sigma} j}. \tag{2.21}$$

We know how to sum finite geometric sums. Indeed, if we call $M = \lfloor R^{1-2\sigma}\rfloor + 1$, we have

$$\sum_{j=1}^{\lfloor R^{1-2\sigma}\rfloor} e^{-2\pi i t R^{\sigma} j} = \frac{e^{-2\pi i t R^{\sigma} M} - 1}{e^{-2\pi i t R^{\sigma}} - 1} - 1 = \frac{e^{-2\pi i t R^{\sigma} M} - e^{-2\pi i t R^{\sigma}}}{e^{-2\pi i t R^{\sigma}} - 1}.$$

It is known that from the subtraction of two exponentials a sine term can be obtained. More precisely,

$$e^{-2\pi i t R^{\sigma}} - 1 = e^{-i\pi t R^{\sigma}}\left(e^{-\pi i t R^{\sigma}} - e^{\pi i t R^{\sigma}}\right) = -2ie^{-i\pi t R^{\sigma}}\sin(\pi R^{\sigma} t)$$

and with the same procedure,

$$e^{-2\pi i t R^{\sigma} M} - e^{-2\pi i t R^{\sigma}} = -2ie^{-i\pi R^{\sigma} t(M-1)}\sin\left(\pi R^{\sigma} t(M+1)\right).$$

Therefore, from (2.21) and knowing that the Fourier transform of ψ is bounded because it is a Schwartz function, we can write

$$\left|\widehat{\eta_R}(t)\right| \leq C R^{2\sigma-1}\frac{|\sin(\pi R^{\sigma} t(M+1))|}{|\sin(\pi R^{\sigma} t)|}$$

and the integral we need to bound in (2.20) becomes

$$\sum_{k\in\mathbb{Z}^n-\{0\}} \frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}} R^{2\sigma-1}\int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^{\sigma} k\cdot\theta(M+1))|}{|\sin(\pi R^{\sigma} k\cdot\theta)|}\,d\theta \tag{2.22}$$

In Lemma 2.8 we show that the integral in (2.22) can be somehow simplified to obtain

$$\int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^{\sigma} k\cdot\theta(M+1))|}{|\sin(\pi R^{\sigma} k\cdot\theta)|}\,d\theta = |\mathbb{S}^{n-2}|\int_{-1}^{1} \frac{|\sin(\pi R^{\sigma}|k|t(M+1))|}{|\sin(\pi R^{\sigma}|k|t)|}(1-t^2)^{(n-3)/2}\,dt,$$

and changing variables $R^{\sigma}|k|t = y$ and bounding $1-t^2 \leq 1$ we get

$$\frac{|\mathbb{S}^{n-2}|}{R^{\sigma}|k|}\int_{-R^{\sigma}|k|}^{R^{\sigma}|k|} \frac{|\sin(\pi(M+1)y)|}{|\sin\pi y|}\,dy. \tag{2.23}$$

Call $N = M+1$ and $A = R^{\sigma}|k|$ for simplicity. We want to analyse

$$\int_{-A}^{A} g_N(t)\,dt, \qquad g_N(t) = \frac{|\sin\pi N t|}{|\sin\pi t|}.$$

Observe that g_N is even and also 1-periodic, since

$$g_N(t+1) = \frac{|\sin(\pi N t + \pi N)|}{|\sin(\pi t + \pi)|} = \frac{|\sin\pi N t|}{|\sin\pi t|} = g_N(t).$$

This allows us to simplify the integral, since

$$\int_{-A}^{A} g_N(t)\,dt = 2\int_0^A g_N(t)\,dt \leq 2\lceil A\rceil\int_0^1 g_N(t)\,dt \approx 2A\int_0^1 g_N(t)\,dt$$

and we can thus work in $(0,1)$. What is more, g_N is even with respect to 1/2, since

$$g_N(1/2+t) = \frac{|\sin(\pi N/2 + \pi N t)|}{|\sin(\pi/2+\pi t)|} = \frac{|\sin(\pi N/2 - \pi N t)|}{|\sin(\pi/2-\pi t)|} = g_N(1/2-t).$$

Therefore,

$$2A\int_0^1 g_N(t)\,dt = 4A\int_0^{1/2} g_N(t)\,dt$$

and we analyse it on $(0,1/2)$, where $\sin\pi t$ only is zero when $t=0$. L'Hôpital's rule asserts that

$$\lim_{t\to 0} g_N(t) = N\lim_{t\to 0}\frac{\cos\pi N t}{\cos\pi t} = N. \tag{2.24}$$

Also, g_N has zeros in $t = k/N$ for $k = 1,2,\dots,$. Our objective is to bound the function in each of the intervals $(k/N,(k+1)/N)$. For that, it is known that

$$\frac{1}{2} \le \frac{\sin x}{x} \le 1, \qquad \forall x \in (0,\pi/2),$$

so

$$g_N(t) = \frac{|\sin\pi N t|}{|\sin\pi t|} \le \frac{1}{|\sin\pi t|} \le \frac{2}{\pi t} \le \frac{2N}{\pi k}, \quad \forall t \in \left(\frac{k}{N},\frac{k+1}{N}\right).$$

Thus, since we are integrating in $(0,1/2)$, we will reach until $k = \lceil N/2\rceil$, and the integral can be bounded by

$$\int_0^{1/2} g_N(t)\,dt \le \frac{1}{N}N + \sum_{k=1}^{\lceil N/2\rceil}\frac{1}{N}\frac{2N}{\pi k} = 1 + \frac{2}{\pi}\sum_{k=1}^{\lceil N/2\rceil}\frac{1}{k}.$$

By comparison of the Riemann sums with the integral, it is easy to see that

$$\sum_{k=1}^{m}\frac{1}{k} \approx \int_1^{m+1}\frac{dx}{x} = \log(m+1),$$

so we obtain

$$\int_0^{1/2} g_N(t)\,dt \le 1 + \frac{2}{\pi}\sum_{k=1}^{\lceil N/2\rceil}\frac{1}{k} \approx 1 + \frac{2}{\pi}\log\lceil N/2\rceil + 1 \lesssim \frac{2}{\pi}\log N.$$

Hence, the integral in (2.23) becomes bounded by

$$\begin{aligned}\frac{|\mathbb{S}^{n-2}|}{R^\sigma|k|}\int_{-R^\sigma|k|}^{R^\sigma|k|}\frac{|\sin(\pi(M+1)y)|}{|\sin\pi y|}\,dy &= \frac{|\mathbb{S}^{n-2}|}{A}\int_{-A}^{A} g_N(t)\,dt \\ &\le \frac{|\mathbb{S}^{n-2}|}{A}4A\int_0^{1/2} g_N(t)\,dt \\ &\lesssim 4|\mathbb{S}^{n-2}|\log N \lesssim \log R^{1-2\sigma},\end{aligned}$$

since recall that $N = M+1 = \lfloor R^{1-2\sigma}\rfloor + 1 \approx R^{1-2\sigma}$. Moreover since $R>1$, then $R^{1-2\sigma} < R$, so we can bound the integral with $\log R$. Hence, (2.22) is bounded by

$$R^{2\sigma-1}\log R\sum_{k\in\mathbb{Z}^n-\{0\}}\frac{R^{-n\sigma}}{(1+R^{-\sigma}|k|)^{n+1}}.$$

We have already seen that the sum remaining is (2.18) and it is summable. Hence, the integral from the beginning we wanted to bound, (2.19), is finally bounded by

$$\left|\int_{\mathbb{R}} \Gamma(x,t,\theta)\eta_R(t)\,dt\right| \lesssim R^{2\sigma-1}\log R$$

for some $\theta \in \mathbb{S}^{n-1}$. Recall that our objective was (2.15), but instead we have obtained also a logarithmic term. Nevertheless, observe that the reasoning works the same if we see that

$$C \le \frac{R^{-n\sigma-2\sigma+1}}{\log R}, \qquad \forall R \gg 1.$$

Call $\alpha = 1-(n+2)\sigma$. By the hypothesis, $\sigma < 1/(n+2)$, so $\alpha > 0$. Now it is enough to see that

$$\lim_{R\to\infty} \frac{R^\alpha}{\log R} = \infty.$$

By L'Hôpital's rule,

$$\lim_{R\to\infty} \frac{R^\alpha}{\log R} = \lim_{R\to\infty} \frac{\alpha R^{\alpha-1}}{1/R} = \lim_{R\to\infty} \alpha R^\alpha = \infty$$

because $\alpha > 0$ and we are done.

□

In the following lines we are to prove the property we used to simplify the integral in (2.22).

Lemma 2.8. *For constants $M,R,\sigma > 0$ and $k \in \mathbb{Z}^n$,*

$$\int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^\sigma (M+1)k\cdot\theta)|}{|\sin(\pi R^\sigma k\cdot\theta)|}\,d\theta = |\mathbb{S}^{n-2}| \int_{-1}^{1} \frac{|\sin(\pi R^\sigma (M+1)|k|t)|}{|\sin(\pi R^\sigma |k|t)|}(1-t^2)^{\frac{n-3}{2}}\,dt,$$

Proof. Observe that we can write

$$k\cdot\theta = |k|\frac{k}{|k|}\cdot\theta.$$

We would like that k had the direction of the canonical vector e_n, so that

$$|k|\frac{k}{|k|}\cdot\theta = |k|e_n\cdot\theta = |k|\theta_n.$$

Observe that the inner product in $\mathbb{R}^n$ is rotation invariant (recall that $k\cdot\theta = |k||\theta|\cos\widehat{k\theta}$), so consider P the rotation sending $k/|k|$ to e_n. We know that $\mathbb{S}^{n-1}$ stays invariant after the rotation, and that its Jacobian is 1, so if f is some function,

$$\int_{\mathbb{S}^{n-1}} |f(k\cdot\theta)|\,d\theta = \int_{\mathbb{S}^{n-1}} |f(P(k)\cdot P(\theta))|\,d\theta = \int_{\mathbb{S}^{n-1}} |f(P(k)\cdot\theta)|\,d\theta,$$

where the last equality corresponds to the change of variables $\theta \to P^{-1}(\theta)$. Hence, we see that we can rotate k and set $k\cdot\theta = |k|\theta_n$ and the original integral is

$$\int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^\sigma (M+1)k\cdot\theta)|}{|\sin(\pi R^\sigma k\cdot\theta)|}\,d\theta = \int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^\sigma (M+1)|k|\theta_n)|}{|\sin(\pi R^\sigma |k|\theta_n)|}\,d\theta.$$

Now we want to work in spherical coordinates. Observe that any point $\theta \in \mathbb{S}^{n-1}$ can be decomposed into an angle and a vector of a lower dimensional sphere:

$$\theta = (\theta_1, \dots \theta_n) \implies \theta_n = \cos\phi_n, \quad (\theta_1, \dots \theta_{n-1}) = u \sin\phi_n,$$

for some $u \in \mathbb{S}^{n-2}$. We can do the same with $u = (u_1, \dots, u_{n-1})$ to obtain $u_{n-1} = \cos\phi_{n-1}$ and $(u_1, \dots, u_{n-2}) = \sin\phi_n v$ for some $v \in \mathbb{S}^{n-3}$. Therefore, it is clear that we can iterate the process to obtain

$$\begin{cases} \theta_n = \cos\phi_n, \\ \theta_{n-1} = \sin\phi_n \cos\phi_{n-1}, \\ \theta_{n-2} = \sin\phi_n \sin\phi_{n-1} \cos\phi_{n-2}, \\ \dots, \\ \theta_2 = \sin\phi_n \sin\phi_{n-1} \dots \sin\phi_3 \cos\phi_2, \\ \theta_1 = \sin\phi_n \sin\phi_{n-1} \dots \sin\phi_3 \sin\phi_2, \end{cases} \tag{2.25}$$

because $\cos\phi_1 = 1$. Hence, the change of variables is

$$\varphi : U = [0, 2\pi] \times [0, \pi] \times \cdots \times [0, \pi] \to \mathbb{S}^{n-1}$$

with $\varphi(\phi_2, \phi_3, \dots, \phi_n) = (\theta_1, \dots, \theta_n)$. This change of variables is indeed a parametrization of a surface in $\mathbb{R}^n$, so we know that we can write

$$\int_{\mathbb{S}^{n-1}} f \, d\sigma = \int_U f(\varphi(y)) \cdot \sqrt{\det\left(\Phi^t \cdot \Phi\right)} \, dy,$$

where Φ is the Jacobian matrix of φ. Computations give

$$\det\left(\Phi^t \cdot \Phi\right) = \left(\sin^2\phi_n\right)^{n-2} \left(\sin^2\phi_{n-1}\right)^{n-3} \dots \left(\sin^2\phi_4\right)^2 \sin^2\phi_3,$$

so call

$$J_n = \sin^{n-2}\phi_n \sin^{n-3}\phi_{n-1} \dots \sin^2\phi_4 \sin\phi_3.$$

If we write the integral, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \frac{|\sin(\pi R^\sigma (M+1)|k|\theta_n)|}{|\sin(\pi R^\sigma |k|\theta_n)|} \, d\theta \\
&\quad = \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \frac{\left|\sin\left(\pi R^\sigma (M+1)|k|\cos\phi_n\right)\right|}{|\sin(\pi R^\sigma |k| \cos\phi_n)|} J_n \, d\phi_n \cdots d\phi_3 \, d\phi_2 \\
&\quad = \int_0^{2\pi} \int_0^{\pi} \cdots \left(\int_0^{\pi} \frac{\left|\sin\left(\pi R^\sigma (M+1)|k|\cos\phi_n\right)\right|}{|\sin(\pi R^\sigma |k| \cos\phi_n)|} \sin^{n-2}\phi_n \, d\phi_n \right) J_{n-1} \, d\phi_{n-1} \cdots d\phi_2 \\
&\quad = \int_0^{\pi} \frac{\left|\sin\left(\pi R^\sigma (M+1)|k|\cos\phi_n\right)\right|}{|\sin(\pi R^\sigma |k| \cos\phi_n)|} \sin^{n-2}\phi_n \, d\phi_n \int_{\mathbb{S}^{n-2}} d\sigma.
\end{aligned}$$

Thus are only left with a one-dimensional integral, because the right-hand side integral is nothing but $|\mathbb{S}^{n-2}|$. Hence, call $t = \cos\phi_n$, so that $\sin\phi_n = \sqrt{1-\cos^2\phi_n} = \sqrt{1-t^2}$. Also, $dt = -\sin\phi_n d\phi_n$, so by this change of variables we obtain

$$
\begin{aligned}
|\mathbb{S}^{n-2}| \int_0^{\pi} &\frac{|\sin\big(\pi R^{\sigma}(M+1)|k|\cos\phi_n\big)|}{|\sin(\pi R^{\sigma}|k|\cos\phi_n)|} \sin^{n-2}\phi_n \, d\phi_n \\
&= |\mathbb{S}^{n-2}| \int_{-1}^{1} \frac{|\sin(\pi R^{\sigma}(M+1)|k|t)|}{|\sin(\pi R^{\sigma}|k|t)|} (1-t^2)^{\frac{n-3}{2}} \, dt,
\end{aligned}
$$

which is what we wanted to prove.

□

Once we have proven the auxiliary lemma, we are ready to tackle the proof of Theorem 2.5.

Proof of Theorem 2.5. Choose $R > 1$. Observe that we can clearly reduce the time interval to $(0, 1/2\pi R)$, since

$$
\sup_{0<t<1} \big| e^{i\frac{t}{2\pi R}\Delta} f \big| = \sup_{0<t<\frac{1}{2\pi R}} \big| e^{it\Delta} f \big| \le \sup_{0<t<1} \big| e^{it\Delta} f \big|,
$$

because $2\pi R > 1$. Also, observe that if we let $\operatorname{supp} \widehat{f} \subset B(0, 2R)$, then several dilations show that

$$
\begin{aligned}
\|f\|^2_{H^s(\mathbb{R}^n)} &= \int_{B(0,2R)} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi = \int_{B(0,2)} |\widehat{f}(R\xi)|^2 (1+R^2|\xi|^2)^s R^n \, d\xi \\
&< R^{2s} \int_{B(0,2)} |\widehat{f}(R\xi)|^2 (1+|\xi|^2)^s R^n \, d\xi \qquad (2.26) \\
&= R^{2s} \int_{B(0,2R)} |\widehat{f}(\xi)|^2 \left(1 + \frac{|\xi|^2}{R^2}\right)^s d\xi \le 5^s R^{2s} \|f\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

where in the last inequality we have noticed that $|\xi|^2 \le 4R^2$ and used Plancherel's identity. Hence, for functions with Fourier transform supported in $B(0, 2R)$ and with $R > 1$, we get

$$
\Big\| \sup_{0<t<1} \big| e^{i\frac{1}{2\pi R}\Delta} f \big| \Big\|_{L^2(B(0,1))} \le C_s R^s \|f\|_{L^2(\mathbb{R}^n)}. \qquad (2.27)
$$

If we are able to prove $s \ge \frac{1}{2} - \frac{1}{n+2}$ using (2.27), we will be done. For this, we will find a particular function, so that applying the new hypothesis (2.27) to it we will obtain the result we desire.

Let $0 < \sigma < \frac{1}{n+2}$ and define

$$
\Omega = \left\{ \xi \in R^{1-\sigma}\mathbb{Z}^n : |\xi| < R \right\} + B(0, \rho),
$$

where $\rho > 0$ will be a small constant which will later be determined. Observe that Ω is nothing but a grid with separation $R^{1-\sigma}$ inside the ball $B(0,R)$ whose intersection points have become small balls of radius ρ. Observe that since the measures of a ball of radius R and a cube of side length $2R$ are comparable, we can estimate the number of grid points inside $B(0,R)$ by looking at the number that lie in $[0,2R]^n$. Indeed, since the separation is $R^{1-\sigma}$, there are $(2R^{\sigma})^n$ grid points in the cube. Hence, there are $\approx R^{n\sigma}$ grid points inside Ω and

$$
|\Omega| \approx |B(0,\rho)| R^{n\sigma} \approx \rho^n R^{n\sigma}. \qquad (2.28)
$$

We take the function whose Fourier transform is the characteristic function of this newly defined set,

$$\widehat{f}(\xi) = \frac{1}{\sqrt{\Omega}} \chi_{\Omega}(\xi).$$

Consider also a direction $\theta \in \mathbb{S}^{n-1}$ and the function f_θ defined as

$$f_\theta(x) = e^{-i\pi R\theta \cdot x} f(x).$$

Observe that at Fourier's side we have made a translation because

$$\widehat{f_\theta}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\pi R\theta\cdot x} e^{2\pi i x\cdot\xi}\, dx = \widehat{f}\left(\xi - \frac{R\theta}{2}\right), \tag{2.29}$$

so $\widehat{f_\theta}$ is supported in $\Omega + \frac{R\theta}{2}$. The above formula shows that by Plancherel's identity,

$$\|f_\theta\|_{L^2} = \|f\|_{L^2} = \frac{1}{\sqrt{\Omega}} \|\chi_\Omega\|_{L^2} = 1, \quad \forall \theta \in \mathbb{S}^{n-1}.$$

We define also for $\epsilon > 0$

$$\Lambda = \left(\{x \in R^{\sigma-1}\mathbb{Z} : |x| < 2\} + B(0, \epsilon R^{-1})\right) \times \{t \in R^{2\sigma-1}\mathbb{Z} : 0 < t < 1\}. \tag{2.30}$$

In what refers to the space variable, Λ is defined in a similar way as Ω. It is a grid of separation $R^{\sigma-1}$ inside the ball of radius 2 whose grid points have been substituted by small balls of radius ϵR^{-1}. On the other hand, we are considering time values in $(0,1)$ with separation $R^{2\sigma-1}$. Our objective is to see that we can bound our solution $e^{i\frac{t}{2\pi R}\Delta} f$ from below when we are lying in this new set Λ. Recall that in the counterexample of Section 1.2, we were looking for certain time values for which the solution was very big. It is the same idea which we are following now. More precisely, we are looking for

$$\sqrt{|\Omega|} \lesssim \left| e^{i\frac{t}{2\pi R}\Delta} f(x) \right|, \quad \forall (x,t) \in \Lambda. \tag{2.31}$$

Observe that by the definition of f,

$$e^{i\frac{t}{2\pi R}\Delta} f(x) = \frac{1}{\sqrt{\Omega}} \int_\Omega e^{2\pi i x\cdot\xi - 2\pi i \frac{t}{R}|\xi|^2}\, d\xi, \tag{2.32}$$

and we want to work with the phase. It is good for us if the phase is very small, since this allows to bound the expression from below by taking only the real part, which is a cosine. For that, we need to analyse the magnitude of $x\cdot\xi$ and $t|\xi|^2/R$. Observe that if $\xi \in \Omega$ and $(x,t) \in \Lambda$, we have

- $\xi = R^{1-\sigma} l + v$, where $l \in \mathbb{Z}^n$, $R^{1-\sigma}|l| < R$ and $|v| < \rho$.
- $x = R^{\sigma-1} m + w$, where $m \in \mathbb{Z}^n$, $R^{\sigma-1}|m| < 2$ and $|w| < \epsilon R^{-1}$.
- $t = R^{2\sigma-1} j$ where $R^{2\sigma-1} j < 1$.

Thus,

$$x \cdot \xi = l \cdot m + R^{1-\sigma} l \cdot w + R^{\sigma-1} m \cdot v + v \cdot w,$$

where $l \cdot m \in \mathbb{Z}$ and

$$|R^{1-\sigma} l \cdot w| \leq \epsilon; \quad |R^{\sigma-1} m \cdot v| \leq 2\rho; \quad |v \cdot w| \leq \epsilon\rho R^{-1}.$$

If we choose $\epsilon, \rho \ll 1$ (for example, $\epsilon, \rho < 1/100$), then

$$\epsilon + 2\rho + \epsilon\rho R^{-1} < \frac{1}{100} + \frac{1}{50} + \frac{1}{10000R} < \frac{1}{20},$$

so we see that $x \cdot \xi \in \mathbb{Z} + B(0, 1/20)$. On the other hand,

$$\frac{t}{R}|\xi|^2 = R^{2\sigma-2} j |R^{1-\sigma} l + v|^2 = j|l|^2 + R^{2\sigma-2} j |v|^2 + 2R^{\sigma-1} j (l \cdot v),$$

where $j|l|^2 \in \mathbb{Z}$ and

$$R^{2\sigma-2} j |v|^2 < R^{-1}\rho^2; \qquad 2R^{\sigma-1} j(l \cdot v) < 2R^{-\sigma} R^{\sigma} \rho = 2\rho.$$

Hence,

$$R^{-1}\rho^2 + 2\rho < \frac{1}{100^2 R} + \frac{2}{100^2} \ll \frac{1}{20},$$

and we can write $\frac{t}{R}|\xi|^2 \in \mathbb{Z} + B(0, 1/20)$. Summing the two terms, we can easily see that

$$x \cdot \xi - \frac{t}{R}|\xi|^2 \in \mathbb{Z} + \left(-\frac{1}{10}, \frac{1}{10}\right),$$

so the phase at (2.32) is very close to 0. The cosine of the phase is thus close to 1, or at least can be bounded by 1/2 from below. Therefore, if we bound the integral by its real part, we get

$$\begin{aligned}
\left| e^{i\frac{t}{2\pi R}\Delta} f(x) \right| &\geq \frac{1}{\sqrt{\Omega}} \left| \int_{\Omega} \cos 2\pi (x \cdot \xi - \frac{t}{R}|\xi|^2) \, d\xi \right| \\
&\geq \frac{1}{\sqrt{\Omega}} \left| \int_{\Omega} \frac{1}{2} \, d\xi \right| = \frac{1}{2}\sqrt{\Omega},
\end{aligned}$$

which is what we wanted to prove as said in (2.31) for points $(x, t) \in \Lambda$.

We want to see if inequality (2.31) can be extended the function f_θ. We first observe that if we explicitly write the solution for f_θ, using (2.29) we see that

$$\begin{aligned}
\left| e^{i\frac{t}{2\pi R}\Delta} f_\theta(x) \right| &= \left| \int_{\mathbb{R}^n} \widehat{f}\left(\xi + \frac{R\theta}{2}\right) e^{2\pi i \left(x \cdot \xi - 2\pi \frac{t}{2\pi R}|\xi|^2\right)} \, d\xi \right| \\
&= \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \left(x \cdot (\xi - \frac{R\theta}{2}) - 2\pi \frac{t}{2\pi R}|\xi - \frac{R\theta}{2}|^2\right)} \, d\xi \right| \\
&= \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \left(x \cdot \xi - 2\pi \frac{t}{2\pi R}|\xi|^2 + \frac{t}{R} R\theta \cdot \xi\right)} \, d\xi \right|,
\end{aligned}$$

showing that

$$\left| e^{i\frac{t}{2\pi R}\Delta} f_\theta(x) \right| = \left| e^{i\frac{t}{2\pi R}\Delta} f(x - t\theta) \right|.$$

Recall that if we want to apply bound (2.31) to f_θ, by what we have seen we need to ask $(x-t\theta,t)\in\Lambda$. Since t needs no changes, looking at the definition of Λ in (2.30) that is the same as asking

$$x\in\{x\in R^{\sigma-1}\mathbb{Z}:|x|<2\}+B(t\theta,\epsilon R^{-1})=\Lambda_{t,\theta}.$$

Therefore, if $x\in\Lambda_{t,\theta}$ and $t\in\{t\in R^{2\sigma-1}\mathbb{Z}:0<t<1\}$, we have

$$\sqrt{|\Omega|}\lesssim\left|e^{i\frac{t}{2\pi R}\Delta}f_\theta(x)\right|.$$

If we take the supremum in t, we see that

$$\sqrt{|\Omega|}\lesssim\sup_{t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)}\left|e^{i\frac{t}{2\pi R}\Delta}f_\theta(x)\right|,\quad\forall x\in\Lambda_\theta=\bigcup_{t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)}\Lambda_{\theta,t}.$$

Moreover, we can also take the supremum in $(0,1)$ and obtain

$$\sqrt{|\Omega|}\lesssim\sup_{t\in(0,1)}\left|e^{i\frac{t}{2\pi R}\Delta}f_\theta(x)\right|,\quad\forall x\in\Lambda_\theta=\bigcup_{t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)}\Lambda_{\theta,t}.$$

Take L^2 norms in Λ_θ to obtain

$$\sqrt{|\Omega||\Lambda_\theta|}\lesssim\Big\|\sup_{t\in(0,1)}\left|e^{i\frac{t}{2\pi R}\Delta}f_\theta\right|\Big\|_{L^2(\Lambda_\theta)}.$$

Recall we said that the support of $\widehat{f_\theta}$ is $\Omega+R\theta/2$, and by the definition of $\xi\in\Omega$,

$$\left|\xi+\frac{R\theta}{2}\right|=\left|y+v+\frac{R\theta}{2}\right|\le|y|+|v|+\frac{R}{2}\le R+\rho+\frac{R}{2}<2R,$$

so $\Omega+R\theta/2\subset B(0,2R)$. Moreover, if $x\in\Lambda_\theta$, then there is some $t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)$ such that $x\in\Lambda_{\theta,t}$ and

$$|x|\le 2+|t|+\epsilon R^{-1}<4,$$

so we can use hypothesis (2.27) to say that

$$\sqrt{|\Omega||\Lambda_\theta|}\lesssim\Big\|\sup_{t\in(0,1)}\left|e^{i\frac{t}{2\pi R}\Delta}f_\theta\right|\Big\|_{L^2(\Lambda_\theta)}\le C_sR^s\|f_\theta\|_{L^2}=C_sR^s. \tag{2.33}$$

We have been able to estimate the measure of Ω, but as we see, we also need to manage the measure of Λ_θ. Observe that if we call

$$E_{\theta,t}=\{x\in R^{\sigma-1}\mathbb{Z}:|x|<2\}+t\theta,$$

then $\Lambda_{\theta,t}$ is a neighbourhood of $E_{\theta,t}$ and for

$$E_\theta=\bigcup_{t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)}E_{\theta,t},$$

then Λ_θ is a neighbourhood of width ϵR^{-1} of E_θ. We know by Lemma 2.7 that there exists some $\theta\in\mathbb{S}^{n-1}$ such that for $R\gg1$, E_θ is ϵR^{-1}-dense in $B(0,1/2)$, which means that the distance from $B(0,1/2)$ to E_θ is less than ϵR^{-1}. This implies that $B(0,1/2)\subset\Lambda_\theta$ and hence

$$2^{-n}\approx|B(0,1/2)|\le|\Lambda_\theta|.$$

Plugging this in (2.33) and recalling (2.28) we get

$$\sqrt{\frac{\rho^n R^{n\sigma}}{2^n}} = \sqrt{|\Omega||\Lambda_\theta|} \lesssim R^s,$$

or what is the same,

$$R^{n\sigma/2} \lesssim R^s$$

for $R \gg 1$. This means that there is a constant $C > 0$ such that $R^{n\sigma/2} \leq CR^s$, or what is the same,

$$\frac{1}{C} \leq R^{s-n\sigma/2}, \quad \forall R \gg 1.$$

This implies that $s - n\sigma/2 > 0$ must hold. Moreover, if we follow the path of the constant along the proof, we will see that it is

$$C = \frac{2C_s 2^{n/2}}{\rho^{n/2}},$$

which can be done as big as we wish by choosing ρ small enough so that $1/C < 1$. In that case, $s - n\sigma/2 = 0$ is also valid, so we deduce that

$$\frac{n\sigma}{2} \leq s,$$

where as we have stated in the beginning, $0 < \sigma < \frac{1}{n+2}$. Then, take $\sigma \to \frac{1}{n+2}$ to obtain

$$\frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)} \leq s.$$

□

3

A Sufficient Condition in $\mathbb{R}^2$

This chapter is devoted to analysing a sufficient condition for our problem (0.7) for the particular case of $n = 2$. The result is due to S. Lee and was presented in [7] in 2006. It is the best known sufficient condition for the two dimensional case so far, which we give in the following theorem explicitly.

Theorem 3.1. *Consider $s > 3/8$ and $f \in H^s(\mathbb{R}^2)$. Then,*

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x)$$

almost everywhere.

By what we have seen in the previous chapters, pointwise convergence arises from a maximal estimate in terms of the Sobolev norm of the initial condition f. Moreover, in various moments we have also checked the effectiveness of proving the estimate for regular Schwartz functions, since the density of these in every H^s space produces, by means of approximation, the result for the whole space. Therefore, our efforts will be again facing the proof of one of these estimates. In what follows in this chapter, unless specified, every function will be considered to be a Schwartz function. The generalisation to H^s will follow by the techniques used in previous chapters.

The proof of Theorem 3.1 requires several mathematical tools and techniques, and it is specially based on the wave-packet decomposition. For this reason, a few first sections will be focused on the development of these needed auxiliary results. We will first present and explain the mentioned wave-packet decomposition as well as a Whitney-type decomposition which will also be useful.

3.1 The Wave-Packet Decomposition

Let $\lambda \gg 1$, for which we define

$$\mathscr{Y} = \lambda^{1/2}\mathbb{Z}^n, \qquad \mathscr{V} = \lambda^{-1/2}\mathbb{Z}^n \cap Q(2),$$

the **space and frequency grids** in $\mathbb{R}^n$, respectively. Here, we denote as $Q(l)$ the cube centred at the origin and of side-length equal to l. Consider any point $(y,v) \in \mathscr{Y} \times \mathscr{V}$, for which we define the **tube** $T_{y,v}$ of dimension $(\lambda^{1/2})^n \times \lambda$ as

$$T_{y,v} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |t| < \lambda, \quad |x-(y+4\pi tv)| \leq \lambda^{1/2}\}.$$

It is easy to see that indeed $T_{y,v}$ is a tube with direction $(2v,1)$ whose centre at $t=0$ is the point y. We call $\mathscr{T}(\lambda)$ to the family of all these tubes, this is to say,

$$\mathscr{T}(\lambda) = \{T_{y,v} \mid (y,v) \in \mathscr{Y} \times \mathscr{V}\}.$$

Hence, we have a bijection $\mathscr{T}(\lambda) \to \mathscr{Y} \times \mathscr{V}$, and for $T \in \mathscr{T}(\lambda)$, if $T = T_{y,v}$, we will call $y(T) = y$ and $v(T) = v$ for simplicity.

Consider now a function $\eta \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\widehat{\eta} \subset Q(2), \qquad \sum_{k\in\mathbb{Z}^n} \eta(\cdot - k) = 1. \tag{3.1}$$

This can be achieved by means of the Poisson summation formula, which asserts that for functions in $\mathscr{S}$,

$$\sum_{k\in\mathbb{Z}^n} f(x+k)e^{-2\pi i(x+k)\cdot\xi} = \sum_{l\in\mathbb{Z}^n} \widehat{f}(\xi+l)e^{2\pi i x\cdot l}.$$

holds pointwise. For $\xi = 0$,

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{l\in\mathbb{Z}^n} \widehat{f}(l)e^{2\pi i x\cdot l},$$

so if we choose $\eta_0 \in \mathscr{S}$ with $\operatorname{supp}\widehat{\eta_0} \subset B(0,1) \subset Q(2)$ and with $\widehat{\eta}_0(0) \neq 0$, we obtain

$$\sum_{k\in\mathbb{Z}^n} \eta_0(x+k) = \widehat{\eta}_0(0) \implies \sum_{k\in\mathbb{Z}^n} \frac{\eta_0(x+k)}{\widehat{\eta}_0(0)} = 1.$$

Hence, the function we seek is $\eta = \eta_0/\widehat{\eta}_0(0)$.

We need also to consider $\psi \in C_0^\infty(B(0,1))$ with $\sum_{k\in\mathbb{Z}^n}\psi(\cdot - k) = 1$. To obtain so, we can consider $\varphi \in C_0^\infty(B(0,2))$ such that $\varphi(x) > 0$ whenever $x \in B(0,1)$ and define

$$\psi(x) = \frac{\varphi(x)}{\sum_{k\in\mathbb{Z}^n}\varphi(x-k)}.$$

The support condition is obviously satisfied, and the sum is always a finite sum also for the support condition. Hence, it is clear that $\sum_{k\in\mathbb{Z}^n}\psi(x-k) = 1, \forall x \in \mathbb{R}^n$.

Using these two recently defined functions, for each $(y,v) \in \mathscr{Y} \times \mathscr{V}$ we define

$$\eta_y(x) = \eta\left(\frac{x-y}{\lambda^{1/2}}\right), \qquad \psi_v(\xi) = \psi(\lambda^{1/2}(\xi - v)). \tag{3.2}$$

Let f be a Schwartz function. Then for each $T \in \mathcal{T}(\lambda)$ we define

$$f_T = \mathcal{F}^{-1}(\widehat{f}\psi_{v(T)})\eta_{y(T)}. \tag{3.3}$$

The following lemma shows that these functions (3.3) form a decomposition for the original function under certain restrictions.

Lemma 3.2. *Let $f \in \mathcal{S}$ be such a function that $\widehat{f}$ is supported in $Q(1)$. Then,*

$$\sum_{T\in\mathcal{T}(\lambda)} f_T = f.$$

Proof. Observe that by the correspondence between the tubes T and the grid points we can write

$$\begin{aligned}\sum_{T\in\mathcal{T}(\lambda)} f_T &= \sum_{T\in\mathcal{T}(\lambda)} \mathcal{F}^{-1}(\widehat{f}\psi_{v(T)})\eta_{y(T)} = \sum_{y\in\mathcal{Y}}\sum_{v\in V} \mathcal{F}^{-1}(\widehat{f}\psi_v)\eta_y \\ &= \sum_{y\in\mathcal{Y}} \eta_y \sum_{v\in V} \mathcal{F}^{-1}(\widehat{f}\psi_v).\end{aligned} \tag{3.4}$$

On the one hand, if $y \in \mathcal{Y}$, then $y\lambda^{-1/2} \in \mathbb{Z}^n$, so

$$\sum_{y\in\mathcal{Y}} \eta_y(x) = \sum_{y\in\mathcal{Y}} \eta\left(\frac{x-y}{\lambda^{1/2}}\right) = \sum_{k\in\mathbb{Z}^n} \eta\left(\lambda^{-1/2}x - k\right) = 1, \tag{3.5}$$

because of the definition of η in (3.1). On the other hand, since $|\mathcal{V}| < \infty$, for the linearity of the Fourier transform, we can write

$$\sum_{v\in V} \mathcal{F}^{-1}(\widehat{f}\psi_v) = \mathcal{F}^{-1}\left(\widehat{f}\sum_{v\in V}\psi_v\right),$$

and

$$\sum_{v\in V}\psi_v(\xi) = \sum_{v\in V}\psi(\lambda^{1/2}(\xi - v)) = \sum_{k\in\mathbb{Z}^n\cap Q(2\lambda^{1/2})} \psi(\lambda^{1/2}\xi - k),$$

because $\lambda^{1/2}v \in \mathbb{Z}^n \cap Q(2\lambda^{1/2})$. But if $\operatorname{supp}\widehat{f} \subset Q(1)$, we are only summing terms such that $\xi \in Q(1)$. Also, for the support of ψ, we need to ask $\lambda^{1/2}\xi - k \in B(0,1)$, which means that $k \in B(\lambda^{1/2}\xi, 1) \subset Q(2\lambda^{1/2})$ because $\lambda \gg 1$. This shows that even if we are summing only some values, the remaining are zero, so

$$\widehat{f}(\xi) \sum_{k\in\mathbb{Z}^n\cap Q(2\lambda^{1/2})} \psi(\lambda^{1/2}\xi - k) = \widehat{f}(\xi)\sum_{k\in\mathbb{Z}^n}\psi(\lambda^{1/2}\xi - k) = \widehat{f}(\xi).$$

Hence, from (3.4) we see that

$$\sum_{T\in\mathcal{T}(\lambda)} f_T = \mathcal{F}^{-1}\widehat{f} = f.$$

□

Lemma 3.2 justifies the definition of the wave-packet decomposition.

Definition 3.3. The decomposition performed in Lemma 3.2 is called the **wave-packet decoposition of f at scale** λ.

It will be important to know the relation between the supports of the original f and each of the wave-packet f_T. The following lemma clarifies so.

Lemma 3.4. *If* $\{f_T\}$ *is the wave packet decomposition of* f *at scale* λ*, then*

$$\operatorname{supp}\widehat{f_T} \subset v(T) + O(\lambda^{-1/2})$$

and also

$$\operatorname{supp}\widehat{f_T} \subset \operatorname{supp}\widehat{f} + O(\lambda^{-1/2}).$$

Proof. From (3.3), the very well-known properties of the Fourier transform assert that

$$\widehat{f_T} = \widehat{f}\psi_v * \widehat{\eta_y},$$

so

$$\operatorname{supp}\widehat{f_T} \subset \operatorname{supp}\widehat{f}\psi_v + \operatorname{supp}\widehat{\eta_y}.$$

First we see that by the definition of ψ_v,

$$\operatorname{supp}\widehat{f}\psi_v \subset \operatorname{supp}\psi_v = \lambda^{-1/2}\operatorname{supp}\psi + v = B(v, \lambda^{-1/2}).$$

On the other hand,

$$\widehat{\eta_y}(\xi) = \mathscr{F}\left[\eta\left(\frac{x-y}{\lambda^{1/2}}\right)\right](\xi) = \lambda^{1/2} e^{-2\pi i \lambda^{1/2}\xi\cdot y}\widehat{\eta}(\lambda^{1/2}\xi),$$

so $\operatorname{supp}\widehat{\eta_y} = \lambda^{-1/2}\operatorname{supp}\widehat{\eta} = Q(2\lambda^{-1/2})$. Thus,

$$\operatorname{supp}\widehat{f_T} \subset B(v, \lambda^{-1/2}) + Q(2\lambda^{-1/2}) \subset B(v, \lambda^{-1/2} + 4\lambda^{-1/2}) = B(v, 5\lambda^{-1/2}).$$

Also, $\operatorname{supp}\widehat{f}\psi_v \subset \operatorname{supp}\widehat{f}$, so

$$\operatorname{supp}\widehat{f_T} \subset \operatorname{supp}\widehat{f} + Q(2\lambda^{-1/2}) = \operatorname{supp}\widehat{f} + O(\lambda^{-1/2}).$$

□

We have already presented the basic properties of the wave-packets. Nevertheless, our interest is not only to decompose the function itself into packets, but to analyse the effect of this decomposition when treating the solution $e^{it\Delta}f$. Indeed, we want to see if we can say something of $e^{it\Delta}f_T$. The main result we are able to give is that each of the packets f_T generates a solution which is mainly concentrated in the tube T.

In what comes, if $m \in C^\infty$, we will denote as $m(D)$ to the multiplier operator given by

$$\mathscr{F}(m(D)f)(\xi) = m(\xi)\widehat{f}(\xi). \tag{3.6}$$

Proposition 3.5. *Let $m \in C^\infty(\mathbb{R}^n)$ be supported in $Q(2)$ and consider $f \in \mathscr{S}(\mathbb{R}^n)$ such that* $\operatorname{supp}\widehat{f} \subset Q(1)$. *Then there exists $C_N = C_N(m) > 0$ such that for $|t| < \lambda$,*

$$|e^{it\Delta} m(D) f_T(x)| \le C_N M(f * \mathscr{F}^{-1}\psi_v)(y)\left(1 + \frac{|x-(y+4\pi t v)|}{\lambda^{1/2}}\right)^{-N} \tag{3.7}$$

for every $N \in \mathbb{N}$ and for every tube $T = T_{y,v}$, where M represent the Hardy-Littlewood maximal function. Also, for any set of tubes $\mathscr{P} \subset \mathscr{T}(\lambda)$ and $|t| \le \lambda$, we have

$$\left\| \sum_{T\in\mathscr{P}} e^{it\Delta} m(D) f_T \right\|_{L^2(\mathbb{R}^n)} \le C\left(\sum_{T\in\mathscr{P}} \|f_T\|_{L^2}^2\right)^{1/2} \le C\|f\|_{L^2}. \tag{3.8}$$

Proof. We have seen in Lemma 3.4 that $\operatorname{supp}\widehat{f_T} \subset B(v, 5\lambda^{-1/2})$. Consider a cutoff function $\widetilde{\psi}$ such that $\widetilde{\psi}_v \equiv 1$ in $\operatorname{supp}\widehat{f_T}$. This can be done by considering a cutoff function φ for $\operatorname{supp}\widehat{f_T}$ and defining $\widetilde{\psi}(\xi) = \varphi(\lambda^{-1/2}\xi + v)$. This way, $\widetilde{\psi}$ is also a cutoff function and $\widetilde{\psi}_v = \varphi$.

Consider the objective at (3.7). By the definition of $m(D)$, we can write

$$e^{it\Delta} m(D) f_T(x) = \int_{\mathbb{R}^n} m(\xi)\widehat{f_T}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi.$$

Since we are integrating in $\operatorname{supp}\widehat{f_T}$, we can introduce $\widetilde{\psi}_v$. Now observe that by the definition of the wave-packet, we can use the Fourier transform properties to write

$$f_T = \mathscr{F}^{-1}(\widehat{f}\psi_{v(T)})\eta_{y(T)} = \left(f * \mathscr{F}^{-1}\psi_v\right)\eta_{y(T)},$$

so writing the definition of the Fourier transform we obtain

$$\begin{aligned} e^{it\Delta} m(D) f_T(x) &= \int_{\mathbb{R}^n} m(\xi)\widetilde{\psi}_v(\xi)\mathscr{F}\left(\left(f * \mathscr{F}^{-1}\psi_v\right)\eta_{y(T)}\right)(\xi) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi \\ &= \int_{\mathbb{R}^n} m(\xi)\widetilde{\psi}_v(\xi)\left(\int_{\mathbb{R}^n} (f * \mathscr{F}^{-1}\psi_v)(z)\eta_{y(T)}(z) e^{-2\pi i \xi\cdot z}\, dz\right) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi \end{aligned} \tag{3.9}$$

We want to change the order of integration. For that, it is necessary that the integral be finite with absolute values in the integrand. Observe that introducing the absolute value generates two independent integrals . One is

$$\int_{\mathbb{R}^n} |m(\xi)\widetilde{\psi}_v(\xi)|\, d\xi,$$

which is finite because we integrate smooth functions with compact supports. The other one is

$$\int_{\mathbb{R}^n} |f * \mathscr{F}^{-1}\psi_v(z)||\eta_y(z)|\, dz.$$

Observe that since $\psi_v \in \mathscr{S}$, then its Fourier transform is so too and hence it is bounded. Thus,

$$|f * \mathscr{F}^{-1}\psi_v(z)| \le \int_{\mathbb{R}^n} |f(z)|\, dz < \infty$$

because $f \in \mathscr{S} \subset L^1$. we only need to worry for the integral of η_y, which is finite because $\eta_y \in \mathscr{S}$ too. Hence, Fubini's theorem allows to change the order of integration in (3.9) to obtain

$$\begin{aligned} e^{it\Delta} m(D) f_T(x) &= \int_{\mathbb{R}^n} (f * \mathscr{F}^{-1}\psi_v)(z)\eta_y(z)\left(\int_{\mathbb{R}^n} m(\xi)\widetilde{\psi}_v(\xi) e^{-2\pi i\xi\cdot z} e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi\right) dz \\ &= \int_{\mathbb{R}^n} (f * \mathscr{F}^{-1}\psi_v)(z)\eta_y(z)\overline{K}(x,t,z)\, dz. \end{aligned}$$

Let us manage the inner integral $\overline{K}$. Since $\widetilde{\psi}_v(\xi) = \widetilde{\psi}(\lambda^{1/2}(\xi - v))$, a change of variables $\xi \to \lambda^{-1/2}\xi + v$ generates

$$\overline{K} = \lambda^{-n/2} \int_{\mathbb{R}^n} m(\lambda^{-1/2}\xi + v)\widetilde{\psi}(\xi) e^{2\pi i((\lambda^{-1/2}\xi+v)\cdot(x-z) - 2\pi t|\lambda^{-1/2}\xi+v|^2)}\, d\xi$$

and we see that $\overline{K}(x,t,z) = \overline{K}(x-z,t)$. Call $\overline{K}(x-z,t) = \lambda^{-n/2}K(x-z,t)$, so that

$$K(x,t) = \int_{\mathbb{R}^n} m(\lambda^{-1/2}\xi + v)\widetilde{\psi}(\xi) e^{2\pi i((\lambda^{-1/2}\xi+v)\cdot x - 2\pi t|\lambda^{-1/2}\xi+v|^2)}\, d\xi \tag{3.10}$$

and

$$e^{it\Delta} m(D) f_T(x) = \lambda^{-n/2} \int_{\mathbb{R}^n} K(x-z,t)\eta\left(\frac{z-y}{\lambda^{1/2}}\right)(f * \mathscr{F}^{-1}\psi_v)(z)\, dz. \tag{3.11}$$

Let us estimate $K(x,t)$. On the one hand, it is easy to see that it is bounded, because the fact that m is bounded and $\widetilde{\psi}$ is smooth and compactly supported implies

$$|K(x,t)| \le \int_{\mathbb{R}^n} |m(\lambda^{-1/2}\xi + v)|\,|\widetilde{\psi}(\xi)|\, d\xi \le \|m\|_\infty \|\widetilde{\psi}\|_1 = C < \infty. \tag{3.12}$$

But on the other hand we observe that K is an oscillatory integral, so we expect that we will be able to give a decaying bound. Its phase is given by

$$A(\xi) = (\lambda^{-1/2}\xi + v)\cdot x - 2\pi t|\lambda^{-1/2}\xi + v|^2,$$

and

$$\nabla A(\xi) = \lambda^{-1/2}(x - 4\pi t v - 4\pi t \lambda^{-1/2}\xi) \tag{3.13}$$

Assume $|x - 4\pi t v| \ge C\lambda^{1/2}$ for some $C \gg 1$. Then we can manage the gradient by means of the triangle inequality to say that

$$\lambda^{1/2}|\nabla A(\xi)| \ge |x - 4\pi t v| - |4\pi t \lambda^{-1/2}\xi|. \tag{3.14}$$

Observe that $|4\pi t\lambda^{-1/2}\xi| \le 4\pi\lambda^{1/2}|\xi|$. Also observe that we are integrating in the support of $\widetilde{\psi}$, and since

$$\widetilde{\psi}(\xi) = \widetilde{\psi}_v(\lambda^{-1/2}\xi + v), \qquad \operatorname{supp}\widetilde{\psi}_v \approx \operatorname{supp}\widehat{f_T} = B(v, 5\lambda^{-1/2}),$$

we see that $\operatorname{supp}\widetilde{\psi} \approx B(0,5)$, so $|4\pi t\lambda^{-1/2}\xi| \le 20\pi\lambda^{1/2}$. By our assumption, and since $C \gg 1$, we see that the second term in (3.14) can be bounded, let us say by

$$|4\pi t \lambda^{-1/2}\xi| < \frac{1}{2}|x - 4\pi t v|,$$

and because of that we obtain

$$\lambda^{1/2}|\nabla A(\xi)| \ge \frac{1}{2}|x - 4\pi t v|.$$

Then, the higher dimensional results for oscillatory integrals in Appendix A.3 assert that

$$\begin{aligned} |K(x,t)| &= \left| \int_{\mathbb{R}^n} m(\lambda^{-1/2}\xi + v)\widetilde{\psi}(\xi) e^{2\pi i A(\xi)}\, d\xi \right| \\ &= \left| \int_{\mathbb{R}^n} m(\lambda^{-1/2}\xi + v)\widetilde{\psi}(\xi) e^{\pi i \lambda^{-1/2}|x-4\pi t v| \frac{2\lambda^{1/2}A(\xi)}{|x-4\pi t v|}}\, d\xi \right| \\ &\le C\left(\pi\lambda^{-1/2}|x - 4\pi t v|\right)^{-N} \end{aligned} \tag{3.15}$$

for every $N \in \mathbb{N}$ and when $|x - 4\pi t v| \geq C\lambda^{1/2}$. Estimates (3.12) and (3.15) allow us to write a general estimate, valid everywhere, which is given by

$$|K(x,t)| \leq C_N \left(1 + 2\pi\lambda^{-1/2}|x - 4\pi t v|\right)^{-N}, \quad \forall N \in \mathbb{N}. \tag{3.16}$$

Now we use (3.16) to develop (3.11). After the change of variables $z \to -z + y$ and calling $a = x - y - 4\pi t v$ and $F(z) = (f * \mathscr{F}^{-1}\psi_v)(-z + y)$, we write

$$\left|e^{it\Delta} m(D) f_T(x)\right| \leq C_N \lambda^{-n/2} \int_{\mathbb{R}^n} \left(1 + \frac{2\pi|a+z|}{\lambda^{1/2}}\right)^{-N} \left|\eta\left(\frac{-z}{\lambda^{1/2}}\right) F(z)\right| dz.$$

If we are able to bound (after renaming $\lambda \to \lambda^2$ for convenience of notation)

$$I = \lambda^{-n} \int_{\mathbb{R}^n} \left(1 + \frac{2\pi|a+z|}{\lambda}\right)^{-N} \left|\eta\left(\frac{-z}{\lambda}\right) F(z)\right| dz \leq C(MF)(0) \left(1 + \frac{|a|}{\lambda}\right)^{-N}, \tag{3.17}$$

then since

$$\begin{aligned} MF(0) &= \sup_{r>0} \fint_{B(0,r)} (f * \mathscr{F}^{-1}\psi_v)(-z+y)\, dz \\ &= \sup_{r>0} \fint_{B(y,r)} (f * \mathscr{F}^{-1}\psi_v)(z)\, dz = M(f * \mathscr{F}^{-1}\psi_v)(y) \end{aligned}$$

we would be done. Hence let us prove (3.17). We need to treat two cases separately: when $|a| \leq \lambda$ and when $|a| > \lambda$.

Assume first that $|a| \leq \lambda$. Observe that this means that $|x - y - 4\pi t v| \leq \lambda$ so the point (x,t) is inside the tube. What we first see is that

$$1 + \frac{2\pi|a+z|}{\lambda} \approx 1 + \frac{|z|}{\lambda}.$$

On the one hand, since $|a| \leq \lambda$ we have

$$1 + \frac{2\pi|a+z|}{\lambda} \leq 1 + \frac{2\pi|a|}{\lambda} + \frac{2\pi|z|}{\lambda} \leq 1 + 2\pi + \frac{2\pi|z|}{\lambda} \leq (1+2\pi)\left(1 + \frac{|z|}{\lambda}\right).$$

Also, by the triangle inequality,

$$1 + \frac{2\pi|a+z|}{\lambda} \geq 1 + \frac{|z|}{\lambda} - \frac{|a|}{\lambda} \geq \frac{|z|}{\lambda},$$

and since it is clearly greater than 1, we have

$$1 + \frac{2\pi|a+z|}{\lambda} \geq \frac{1}{2}\left(1 + \frac{|z|}{\lambda}\right),$$

so the claimed comparability holds. Hence, (3.17) can be transformed into

$$I \approx \lambda^{-n} \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{\lambda}\right)^{-N} \left|\eta\left(\frac{-z}{\lambda}\right) F(z)\right| dz = \int_{\mathbb{R}^n} H\left(\frac{|z|}{\lambda}\right) \left|\eta\left(\frac{-z}{\lambda}\right) F(z)\right| dz.$$

We know that η is a Schwartz function, so it is bounded. Therefore,

$$I \lesssim \lambda^{-n} \int_{\mathbb{R}^n} H\left(\frac{|z|}{\lambda}\right) |F(z)|\, dz = \lambda^{-n} \left(H\left(\frac{\cdot}{\lambda}\right) * F\right)(0).$$

Now we can apply Lemma B.2 to have a bound with the maximal function, so that

$$I \lesssim \lambda^{-n} \left\| H\left(\frac{\cdot}{\lambda}\right) \right\|_{L^1} MF(0).$$

That L^1-norm can be treated by a change of variables $z \to \lambda z$ to obtain

$$\lambda^{-n} \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{\lambda}\right)^{-N} dz = \int_{\mathbb{R}^n} (1+|z|)^{-N}\, dz = C \int_0^\infty \frac{r^{n-1}}{(1+r)^n}\, dr,$$

which we know is finite if $N \geq n+1$. Hence we get

$$I \lesssim CMF(0).$$

But by the case we are in,

$$\frac{|a|}{\lambda} + 1 \leq 2 \quad \Longrightarrow \quad 1 \leq 2^N \left(1 + \frac{|a|}{\lambda}\right)^{-N},$$

so we can conclude by writing

$$I \leq CMF(0) \leq C2^N \left(1 + \frac{|a|}{\lambda}\right)^{-N} MF(0) = C_N \left(1 + \frac{|a|}{\lambda}\right)^{-N} MF(0)$$

as we wished.

The case $|a| \geq \lambda$ is more complicated. We need to split I into two parts. Write

$$I = I_0 + I_1$$

where I_0 integrates in $|z| \leq \lambda/2$ and I_1 in $|z| > \lambda/2$. Let us first analyse I_0. Observe that in this case,

$$|z| \leq \frac{\lambda}{2} \leq \frac{|a|}{2} \implies 1 + 2\pi \frac{|a+z|}{\lambda} \geq \frac{|a| - |z|}{\lambda} \geq \frac{|a|}{2\lambda}.$$

This implies that, knowing that η is bounded, we can write

$$I_0 \leq \lambda^{-n} \left(\frac{|a|}{2\lambda}\right)^{-N} \int_{|z|<\lambda/2} |F(z)|\, dz = 2^{N-n} \left(\frac{|a|}{\lambda}\right)^{-N} \fint_{|z|<\lambda/2} |F(z)|\, dz.$$

The integral can of course be bounded by the maximal function, so we obtain

$$I_0 \leq C_N \left(\frac{|a|}{\lambda}\right)^{-N} MF(0). \tag{3.18}$$

This is not precisely the bound we want, but it will not be a big problem as we will later see. Let us focus now on obtaining a bound for I_1. In this case we do not have a ball, so we will split the outer integral into dyadic annuli to be able to manage them by means of the maximal function. Write

$$I_1 = \lambda^{-n} \sum_{j=0}^{\infty} \int_{|z| \approx \lambda 2^j} \left(1 + \frac{2\pi|a+z|}{\lambda}\right)^{-N} \left| \eta\left(\frac{-z}{\lambda}\right) F(z) \right| dz.$$

Since $|a| > \lambda$, we can find the annulus in which it lies, this is to say, $|a| \approx \lambda 2^k$ for some $k \in \mathbb{N}$. We will treat each integral separately.

First we suppose $j = k$. This means that $|a| \approx |z| \approx \lambda 2^j$. In this case, we will make us of the rapid descent of η. Indeed, for any $M \in \mathbb{N}$, $|z|^M |\eta(z)| < C_M$, so

$$\left|\eta\left(\frac{-z}{\lambda}\right)\right| \le C_M \left(\frac{|z|}{\lambda}\right)^{-M} \approx C_M \left(\frac{|a|}{\lambda}\right)^{-M}, \qquad \forall M \in \mathbb{N}.$$

Hence, if we simply bound the $-N$ power term by 1, we get a bound for the $\lambda 2^k$-annulus integral of

$$\begin{aligned} \lambda^{-n} \int_{|z|\approx\lambda 2^j} & \left(1 + \frac{2\pi|a+z|}{\lambda}\right)^{-N} \left|\eta\left(\frac{-z}{\lambda}\right) F(z)\right| dz \\ & \le \lambda^{-n} C_M \left(\frac{|a|}{\lambda}\right)^{-M} \int_{|z|\approx\lambda 2^j} |F(z)|\, dz \\ & \le C_M \left(\frac{|a|}{\lambda}\right)^{-M} 2^{jn} \fint_{|z|\le\lambda 2^j} |F(z)|\, dz \\ & \le C_M \left(\frac{|a|}{\lambda}\right)^{-M} 2^{jn} MF(0). \end{aligned}$$

Recall that $|a| \approx \lambda 2^j$, so the bound becomes

$$C_M \left(\frac{|a|}{\lambda}\right)^{-M+n} MF(0), \quad \forall M \in \mathbb{N},$$

and since n is the fixed dimension, we get

$$C_M \left(\frac{|a|}{\lambda}\right)^{-M} MF(0), \quad \forall M \in \mathbb{N}. \tag{3.19}$$

When $j \ne k$, we need to use the previously discarded term too, since the decreasing properties of η are not enough this time. Recall that the term $|a|/\lambda$ was obtained through η because $|z| \approx |a|$. Now we cannot do that. Nevertheless, observe that

- When $j < k$, we have $|a| \approx \lambda 2^k > \lambda 2^j \approx |z|$, so

  $$|a+z| \ge |a| - |z| \approx \lambda 2^k - \lambda 2^j = \lambda 2^k (1 - 2^{j-k}) \approx |a| C_j,$$

  where $C_j = 1 - 2^{j-k} \ge 1 - 2^{-1} = 1/2$. Therefore, $|a+z| \ge |a|/2$.

- When $j > k$, we have $|a| \approx \lambda 2^k < \lambda 2^j \approx |z|$, and in this case,

  $$|a+z| \ge |z| - |a| \approx \lambda(2^j - 2^k) \approx |a|(2^{j-k} - 1) \ge |a|.$$

In any case, we see that $|a| \le 2|a+z|$, so we can say that

$$\left(1 + 2\pi\frac{|a+z|}{\lambda}\right)^{-N} \le \left(\frac{|a+z|}{\lambda}\right)^{-N} \le \left(\frac{|a|}{2\lambda}\right)^{-N} = 2^N \left(\frac{|a|}{\lambda}\right)^{-N}$$

and we have obtained the desired $|a|/\lambda$ term. Now going back to the integral and introducing also the same bound for η as before, we have

$$\begin{aligned}\lambda^{-n}\int_{|z|\approx\lambda 2^j}&\left(1+\frac{2\pi|a+z|}{\lambda}\right)^{-N}\left|\eta\left(\frac{-z}{\lambda}\right)F(z)\right|dz\\&\le\lambda^{-n}2^N\left(\frac{|a|}{\lambda}\right)^{-N}C_M\int_{|z|\approx\lambda 2^j}\left(\frac{|z|}{\lambda}\right)^{-M}|F(z)|\,dz\\&\lesssim C_M2^N\left(\frac{|a|}{\lambda}\right)^{-N}2^{-jM}2^{jn}⨍_{|z|\approx\lambda 2^j}|F(z)|\,dz\\&\le C_M2^N\left(\frac{|a|}{\lambda}\right)^{-N}2^{j(n-M)}MF(0).\end{aligned}\tag{3.20}$$

Thus, we have obtained every bound we wanted. Let us join everything. From (3.18), (3.19) and (3.20) we have (observing that the first two have given the same bound)

$$\begin{aligned}I=I_0+I_1&\le C_M\left(\frac{|a|}{\lambda}\right)^{-M}MF(0)+\sum_{j\in\mathbb{N}-\{k\}}C_M2^N\left(\frac{|a|}{\lambda}\right)^{-N}2^{j(n-M)}MF(0)\\&\le C_N\left(\frac{|a|}{\lambda}\right)^{-N}MF(0)+C_M2^N\left(\frac{|a|}{\lambda}\right)^{-N}MF(0)\sum_{j\in\mathbb{N}-\{k\}}2^{j(n-M)}.\end{aligned}$$

Since it is valid for every $M\in\mathbb{N}$, choose M big enough so that $M>n$. Hence, the sum converges and we see that

$$I\le C_N\left(\frac{|a|}{\lambda}\right)^{-N}MF(0).\tag{3.21}$$

We are almost done, since we are missing a unity inside the power. But observe that $|a|\ge\lambda$ implies directly from (3.21) that the bound is also

$$I\le C(MF)(0),\tag{3.22}$$

a constant in this case. It is a known result that two bounds as (3.21) and (3.22) imply

$$I\le C_N\left(1+\frac{|a|}{\lambda}\right)^{-N}MF(0),$$

which is what we asked for in (3.17) to prove in (3.7).

Hence let us prove the second part of the proposition, (3.8). Recall that we saw in Lemma 3.2 that

$$f=\sum_{T\in\mathcal{T}(\lambda)}f_T=\sum_{v\in\mathcal{V}}\sum_{y\in\mathcal{Y}}f_{T_{y,v}}.$$

For each $v\in\mathcal{V}$ call $f_v=\sum_{y\in\mathcal{Y}}f_{T_{y,v}}$. Thus, $f=\sum_{v\in\mathcal{V}}f_v$ and we observe by Plancherel that,

$$\|f\|_2^2=\left\|\mathcal{F}\Big(\sum_{v\in\mathcal{V}}f_v\Big)\right\|_2^2=\Big\|\sum_{v\in\mathcal{V}}\widehat{f_v}\Big\|_2^2,\tag{3.23}$$

because the sum in $\mathcal{V}$ is finite. We would like to take the sum out of the norm, but that cannot be done by the triangle inequality directly. Let us use the properties of the decomposition instead.

Observe that

$$f_v = \sum_{y\in\mathscr{Y}} f_{T_{y,v}} = \sum_{y\in\mathscr{Y}} \mathscr{F}^{-1}(\widehat{f}\psi_v)\eta_y = \mathscr{F}^{-1}(\widehat{f}\psi_v)\sum_{y\in\mathscr{Y}} \eta_y = \mathscr{F}^{-1}(\widehat{f}\psi_v),$$

for the property we saw in (3.5). This shows that $\widehat{f_v} = \widehat{f}\psi_v$. Therefore, $\operatorname{supp}\widehat{f_v} \subset \operatorname{supp}\psi_v = B(v,\lambda^{-1/2})$. This allows us to say by Cauchy-Schwartz that

$$\Big|\sum_{v\in\mathscr{V}} \widehat{f_v}(\xi)\Big|^2 = \Big|\sum_{v\in\mathscr{V}} \chi_{B(v,\lambda^{-1/2})}(\xi)\widehat{f_v}(\xi)\Big|^2 \le \sum_{v\in\mathscr{V}} \chi_{B(v,\lambda^{-1/2})}(\xi)^2 \sum_{v\in\mathscr{V}} |\widehat{f_v}(\xi)|^2.$$

Observe now that the frequency grid $\mathscr{V}$ has separation $\lambda^{-1/2}$, and thus the sum of the characteristic function is at most the sum of 2^n of them for each point. Hence, we can write

$$\Big|\sum_{v\in\mathscr{V}} \widehat{f_v}(\xi)\Big|^2 \le 2^n \sum_{v\in\mathscr{V}} |\widehat{f_v}(\xi)|^2. \tag{3.24}$$

We want to obtain a reverse inequality too. For that, sum in $\mathscr{V}$ to obtain

$$\sum_{v\in\mathscr{V}} |\widehat{f_v}(\xi)|^2 = |\widehat{f}(\xi)|^2 \sum_{v\in\mathscr{V}} |\psi_v(\xi)|^2. \tag{3.25}$$

We need to manage this sum. Indeed,

$$\sum_{v\in\mathscr{V}} |\psi_v(\xi)|^2 = \sum_{k\in\mathbb{Z}^n\cap Q(2\lambda^{1/2})} |\psi(\lambda^{1/2}\xi - k)|^2,$$

and an auxiliary lemma will give us the way to treat it.

Lemma 3.6. *Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\sum_{k\in\mathbb{Z}^n} |\phi(x-k)| = 1$. Then, there exist constants C_1, C_2 such that*

$$0 < C_1 < \sum_{k\in\mathbb{Z}^n} |\phi(x-k)|^2 < C_2 < \infty. \tag{3.26}$$

Proof of Lemma 3.6. Call $F(x) = \sum_{k\in\mathbb{Z}^n} |\phi(x-k)|^2$, which is clearly 1-periodic. By the hypothesis regarding the sum, it is obvious that $F(x) \neq 0, \forall x \in \mathbb{R}^n$. If we are able to show its continuity, it will attain both a maximum and a minimum. Moreover, the minimum will not be zero, since the function itself does not vanish. Hence it is enough to prove the continuity of F. By the periodicity it is enough to prove it in $[0,1]^n$. So consider $\epsilon > 0$, $x \in [0,1]^n$ and $y \in \mathbb{R}^n$. We write

$$|F(x) - F(y)| \le \Big|\sum_{|k|\le A} |\phi(x-k)|^2 - |\phi(y-k)|^2\Big| + \Big|\sum_{|k|>A} |\phi(x-k)|^2 - |\phi(y-k)|^2\Big|$$

for some $A > 0$. We see that

$$\Big|\sum_{|k|>A} |\phi(x-k)|^2 - |\phi(y-k)|^2\Big| \le \sum_{|k|>A} |\phi(x-k)|^2 + \sum_{|k|>A} |\phi(y-k)|^2,$$

and by the Schwartz property, any $M \in \mathbb{N}$ gives

$$\sum_{|k|>A} |\phi(y-k)|^2 \le C_M \sum_{|k|>A} \frac{1}{|y-k|^M}.$$

Now, if y is near x, let us say $|x-y|<1$, then $|y-k|>|k|-|y|>|k|/2$ because $|k|>A\gg 1$. Hence, we choose $A\gg 1$ and M much bigger than the dimension so that

$$\sum_{|k|>A}|\phi(y-k)|^2\le C_M\sum_{|k|>A}\frac{2^M}{|k|^M}<\epsilon/4.$$

The same works for the case of x, so the sum in $|k|>A$ is bounded by $\epsilon/2$. Now, the sum in $|k|\le A$ is finite, so it is continuous, so there exists $\delta>0$ so that $|x-y|<\delta$ implies $\sum_{|k|\le A}|\phi(y-k)|^2<\epsilon/2$ and we are done. □

Coming back to (3.25), we can bound it with the sum in the whole integer grid to have

$$\sum_{v\in\mathcal{V}}|\widehat{f}_v(\xi)|^2\le C_2|\widehat{f}(\xi)|^2=C_2\Big|\sum_{v\in\mathcal{V}}\widehat{f}_v(\xi)\Big|^2. \tag{3.27}$$

Then, joining (3.24) and (3.27) we obtain our first important result,

$$\sum_{v\in\mathcal{V}}|\widehat{f}_v(\xi)|^2\approx\Big|\sum_{v\in\mathcal{V}}\widehat{f}_v(\xi)\Big|^2. \tag{3.28}$$

This implies together with (3.23) that we have

$$\|f\|_2^2=\Big\|\sum_{v\in\mathcal{V}}\widehat{f}_v\Big\|_2^2=\int\Big|\sum_{v\in\mathcal{V}}\widehat{f}_v(\xi)\Big|^2d\xi\approx\int\sum_{v\in\mathcal{V}}|\widehat{f}_v(\xi)|^2\,d\xi=\sum_{v\in\mathcal{V}}\|f_v\|_2^2. \tag{3.29}$$

Finally, we need to manage $\|f_v\|_2^2$. We know that

$$\|f_v\|_2^2=\Big\|\sum_{y\in\mathcal{Y}}f_{T_{y,v}}\Big\|_2^2=\int\Big|\sum_{y\in\mathcal{Y}}f_{T_{y,v}}(\xi)\Big|^2d\xi.$$

Again we want to take the sum out of the integral to obtain

$$\int|\sum_{y\in\mathcal{Y}}f_T(\xi)|^2\approx\sum_{y\in\mathcal{Y}}\int|f_T(\xi)|^2=\sum_{y\in\mathcal{Y}}\|f_T\|_2^2. \tag{3.30}$$

For that, notice that since we are working with positive functions we can write

$$\sum_{y\in\mathcal{Y}}\int|f_T|^2=\int\sum_{y\in\mathcal{Y}}|f_T|^2=\int|\mathcal{F}^{-1}(\widehat{f}\psi_v)|^2\sum_{y\in\mathcal{Y}}|\eta_y|^2\approx\int|\mathcal{F}^{-1}(\widehat{f}\psi_v)|^2. \tag{3.31}$$

In the last step we have made use of Lemma 3.6. Now by Plancherel that last integral is clearly $\|\widehat{f}\psi_v\|_2^2$. On the other hand,

$$\int|\sum_{y\in\mathcal{Y}}f_T|^2=\int|\mathcal{F}^{-1}(\widehat{f}\psi_v)|^2|\sum_{y\in\mathcal{Y}}\eta_y|^2=\int|\mathcal{F}^{-1}(\widehat{f}\psi_v)|^2=\|\widehat{f}\psi_v\|_2^2, \tag{3.32}$$

so joining (3.31) and (3.32) we get (3.30) as we desired. In short, we have proven that

$$\|f\|_2^2=\|\sum_{T\in\mathcal{T}(\lambda)}f_T\|_2^2\approx\sum_{v\in\mathcal{V}}\|f_v\|_2^2\approx\sum_{v\in\mathcal{V}}\sum_{y\in\mathcal{Y}}\|f_T\|_2^2=\sum_{T\in\mathcal{T}(\lambda)}\|f_T\|_2^2. \tag{3.33}$$

This is one of the main advantages of working with wave-packets: we have **a version of Plancherel's identity**. Also observe that the equivalences above are still true if we choose any $\mathscr{P} \subset T(\lambda)$. Hence, for any $\mathscr{P} \subset \mathscr{T}(\lambda)$ (3.33) implies

$$\sum_{T\in\mathscr{P}} \|f_T\|_2^2 \leq \sum_{T\in\mathscr{T}(\lambda)} \|f_T\|_2^2 \approx \|f\|_2^2$$

which is the second inequality in (3.8).

Finally, to obtain the first inequality in (3.8) we only need to take into account that $e^{it\Delta}$ is a linear isometry in L^2. The multiplier $m(D)$ is also linear, so

$$\Big\|\sum_{T\in\mathscr{P}} e^{it\Delta} m(D) f_T\Big\|_{L^2(\mathbb{R}^n)} = \Big\|e^{it\Delta} m(D) \sum_{T\in\mathscr{P}} f_T\Big\|_{L^2(\mathbb{R}^n)} = \Big\|m(D) \sum_{T\in\mathscr{P}} f_T\Big\|_{L^2(\mathbb{R}^n)}.$$

Now, by Plancherel we can change to the Fourier space, and since the effect of $m(D)$ is to multiply $m(\xi)$, we bound it to obtain

$$\Big\|m(D) \sum_{T\in\mathscr{P}} f_T\Big\|_{L^2(\mathbb{R}^n)} \leq \|m\|_\infty \Big\|\sum_{T\in\mathscr{P}} f_T\Big\|_{L^2(\mathbb{R}^n)}.$$

Finally, the very useful (3.33) asserts that

$$\Big\|\sum_{T\in\mathscr{P}} e^{it\Delta} m(D) f_T\Big\|_{L^2(\mathbb{R}^n)}^2 \leq C\|m\|_\infty^2 \sum_{T\in\mathscr{P}} \|f_T\|_2^2.$$

□

3.2 A Whitney-type Decomposition

In Section 3.1 we analysed the decomposition of a function into waves, obtaining a wave-packet which satisfies very interesting properties regarding the Schrödinger solution $e^{it\Delta}f$. When we talk about a Whitney-type decomposition, though, the concept changes since we are looking for decomposing the space instead of functions. In our case, we will need to decompose the space $\mathbb{R}^n \times \mathbb{R}^n$ in the way we are about to see. The idea is to split the space into dyadic cubes of varying size so that the closer we stay from the main diagonal $\Gamma = \{(x,x) \mid x \in \mathbb{R}^n\}$ the smaller the cubes are.

The main ingredient needed is a classification of any two points $\xi, \eta \in \mathbb{R}^n$ into certain dyadic cubes. We present it in the form of the following lemma.

Lemma 3.7. *Let $\xi, \eta \in \mathbb{R}^n$. Then, there exist Q_1, Q_2 unique dyadic cubes such that*

- $\xi \in Q_1$ *and* $\eta \in Q_2$,
- $l(Q_1) = l(Q_2)$, *and*
- $l(Q_1) \leq \operatorname{dist}(Q_1, Q_2) \leq 4\sqrt{n}\, l(Q_1)$

where $l(Q)$ denotes the side-length of the cube Q.

Proof. The idea is very geometric. What we need to consider is the smallest dyadic size 2^j in which the cubes containing ξ and η are adjacent. Therefore, consider Q_1 and Q_2 the cubes of size 2^{j-1} containing ξ and η respectively. By the choice of their parents, Q_1 and Q_2 are not adjacent and

$$l(Q_1) = 2^{j-1} = l(Q_2).$$

Moreover, since they are nor adjacent, there must be some other cube between them, showing that

$$\operatorname{dist}(Q_1, Q_2) \geq 2^{j-1}.$$

Now, since the two parents of size 2^j are adjacent , they can be included in a cube of size 2^{j+1}, in which both Q_1, Q_2 must lie. Therefore, the distance between them cannot be greater than the diagonal of this big cube, so

$$\operatorname{dist}(Q_1, Q_2) \leq \operatorname{diag}\left(Q(2^{j+1})\right) = \sqrt{n}2^{j+1} = 4\sqrt{n}2^{j-1}.$$

The points, for being in Q_1 and Q_2, will be at a distance trivially comparable to the distance between the cubes. Hence, we see that

$$l(Q_1) = l(Q_2) = 2^{j-1} \approx \operatorname{dist}(Q_1, Q_2) \approx |\xi - \eta|,$$

where the comparability constants are 1 and $4\sqrt{n}$ which only depend on the dimension. □

Lemma 3.7 is the result which will allow us to build a decomposition of the space save the main diagonal,

$$\mathbb{R}^n \times \mathbb{R}^n - \Gamma.$$

Indeed, for $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ we have checked the existence of dyadic cubes Q_1, Q_2 of length $\approx \operatorname{dist}(\xi, \eta)$. This means that if we consider $(\xi, \eta) \in Q = Q_1 \times Q_2$, the further the point is from Γ, the bigger the cube will be. Analytically,

- we denote as τ_k^j a dyadic cube of length 2^{-j}, and
- we denote $\tau_k^j \sim \tau_{k'}^j$ if they are not adjacent but their parents are so.

Observe that the cubes chosen in Lemma 3.7 are related by the definition we have just given. Therefore, we can decompose the space as

$$\mathbb{R}^n \times \mathbb{R}^n \setminus \Gamma = \bigcup_{j \in \mathbb{Z}} \bigcup_{\tau_k^j \sim \tau_{k'}^j} \tau_k^j \times \tau_{k'}^j. \tag{3.34}$$

Definition 3.8. The decomposition given by (3.34) is called a **Whitney-type decomposition** of the space away from its diagonal.

We will refer to this definition when we tackle the proof of the main result of this chapter.

3.3 Some Auxiliary Lemmas

Once we have the wave-packet decomposition for functions and a Whitney-type decomposition of the space, we will here present some auxiliary lemmas which play a key role in the proof of Theorem 3.1. Most of these lemmas will be, as we will see, applications of the wave-packet decomposition detailed in Section 3.1.

We first present a basic lemma which will be of great help in many steps. For the sake of concision, we will only give the main steps of the proof.

Lemma 3.9. *Let F be a smooth function on an interval $[a,b]$. Then, there is $C>0$ so that for any $\mu>0$ we have*

$$\sup_{t\in[a,b]} |F(t)| \le C\left(|F(a)| + \mu^{1/2}\|F\|_{L^2([a,b])} + \mu^{-1/2}\|F'\|_{L^2([a,b])}\right).$$

Proof. Since the derivative of F^2 is $2FF'$, we write $\int_a^t 2F(t)F'(t)\,dt = F^2(t) - F^2(a)$, so

$$|F(t)|^2 \le |F(a)|^2 + 2\int_a^b |F(t)F'(t)|\,dt.$$

After writing the harmless $1=\mu^r\mu^{-r}$ and by Young's inequality we have

$$|F(t)|^2 \le |F(a)|^2 + \mu^{2r}\|F\|_2^2 + \mu^{-2r}\|F'\|_2^2,$$

and if we root that expression, we have

$$\begin{aligned}|F(t)| &\le \left(|F(a)|^2 + \mu^{2r}\|F\|_2^2 + \mu^{-2r}\|F'\|_2^2\right)^{1/2}\\ &\le \sqrt{3}\left(|F(a)| + \mu^{r}\|F\|_2 + \mu^{-r}\|F'\|_2\right).\end{aligned}$$

The result is obtained by choosing $r=1/2$. □

The two following lemmas we are going to see have more visible and direct importance in the proof of the final result. They are a consequence of the wave-packet decomposition analysed in Section 3.1, and they give us the way to decompose the initial data f into functions f_j so that $e^{it\Delta}f$ is basically $e^{it\Delta}f_j$ when we restrict ourselves to certain regions in space or in time. Let us state them precisely. To clarify notation, as we have used before, we write $Q(\lambda)$ for the cube centred at the origin and of side-length λ and $A(\lambda)$ to denote the annulus of radii $\lambda/2$ and λ.

Lemma 3.10. *Let $\lambda \gg 1$ and consider a partition of the time interval $[0,\lambda^2]$ given by intervals I_j of length $\approx\lambda$. Consider also a function f such that $\operatorname{supp}\widehat{f}\subset A(1)$. Then, for any $\epsilon>0$, there exist functions f_j whose Fourier transform is supported in $\operatorname{supp}\widehat{f}+O(\lambda^{-1})$ such that*

$$\left(\sum_j \|f_j\|_2^2\right)^{1/2} \le C_\epsilon \lambda^\epsilon \|f\|_2$$

and also

$$|e^{it\Delta}f(x)| \le |e^{it\Delta}f_j(x)| + C\lambda^{-N}\|f\|_2, \qquad \forall N\in\mathbb{N}$$

whenever $x\in Q(\lambda)$ and $t\in I_j$.

Lemma 3.10 represents the idea explained above in the sense that the original time interval $[0,\lambda^2]$ can be partitioned into smaller intervals I_j where $e^{it\Delta}f$ can be basically substituted by $e^{it\Delta}f_j$. The idea underlying Lemma 3.11 is very similar, since it states that we can also partition the original space into smaller cubes where the whole solution can be managed through an auxiliary function.

Lemma 3.11. *Let $\lambda, M \gg 1$ and consider a partition of the cube $Q(M\lambda)$ into cubes Q_l of side-length $\approx \lambda$. Let f be a function whose Fourier transform is supported in $Q(1)$. Then, for any $\epsilon > 0$, there are functions f_l with Fourier support in* $\operatorname{supp}\widehat{f} + O(\lambda^{-1/2})$ *such that*

$$\left(\sum_l \|f_l\|_2^2\right)^{1/2} \le C_\epsilon \lambda^\epsilon \|f\|_2$$

and

$$|e^{it\Delta}f(x)| \le |e^{it\Delta}f_l(x)| + C\lambda^{-N}\|f\|_2, \qquad \forall N \in \mathbb{N}$$

for points $x \in Q_l$ and time $|t| < \lambda$.

The decision to state both lemmas at the same time is reasonable on the one hand for their visible similarity, but also because their proofs are basically the same. Here we will give the proof of Lemma 3.10 with every detail, and we will comment the points which need a little change for the proof of Lemma 3.11.

Proof of Lemma 3.10. It is clear that by the partition made in the statement, there will be $\approx \lambda$ intervals I_j. We denote the spacial region for each of these intervals as

$$q_j = Q(\lambda) \times I_j.$$

Consider the wave-packet decomposition for f at scale λ^2 (even if $\operatorname{supp}\widehat{f}$ is not in $Q(1)$, this condition given in Section 3.1 can be modified to bigger cubes by taking a slightly bigger frequency grid $\mathcal{V}$). Observe that in this case, the grids are

$$\mathcal{Y} = \lambda\mathbb{Z}^n, \qquad \mathcal{V} = \frac{1}{\lambda}\mathbb{Z}^n \cap Q(2),$$

and the decomposition is given by $f = \sum_{T\in\mathcal{T}(\lambda^2)} f_T$. We want to use the bound (3.7) at Proposition 3.5. We know that the maximal function is bounded by

$$M(f * \mathcal{F}^{-1}\psi_v)(y) \le \|f * \mathcal{F}^{-1}\psi_v\|_{L^\infty},$$

and by Hölder's inequality we have $|f * \mathcal{F}^{-1}\psi_v| \le \|f\|_2 \|\mathcal{F}^{-1}\psi_v\|_2$. Also a change of variables shows that

$$\|\mathcal{F}^{-1}\psi_v\|_2^2 = \|\psi_v\|_2^2 = \int \psi(\lambda(\xi - v))\, d\xi = \lambda^{-n}\|\psi\|_2^2.$$

Hence,

$$M(f * \mathcal{F}^{-1}\psi_v)(y) \le C\lambda^{-n/2}\|f\|_2$$

and the bound in Proposition 3.5 turns into

$$|e^{it\Delta} f_T(x)| \leq C_N \lambda^{-n/2} \|f\|_2 \left(1 + \frac{|x-(y+4\pi t v)|}{\lambda}\right)^{-N}, \quad \forall N \in \mathbb{N}. \tag{3.35}$$

Recalling that the tubes are $T_{y,v} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |t| < \lambda^2, \quad |x-(y+4\pi t v)| \leq \lambda\}$, we can say that $|x-(y+4\pi t v)|$ is the distance between the point (x,t) and the centre of the tube. Then (3.35) shows that $|e^{it\Delta} f_T(x)|$ decreases as rapidly as we wish outside the corresponding tube T.

We are interested in what happens in $Q(\lambda) \times [0,\lambda^2]$. Because of this, we will select tubes T which intersect $\lambda^\epsilon \left(Q(\lambda) \times [0,\lambda^2]\right)$. The reason for this λ^ϵ term will become clear in the following computations. We want to count the number of tubes $T \in \mathcal{T}(\lambda^2)$ that satisfy this condition (the rest will have no effect). Recall that if $T = T_{y,v}$, then the tube is centred in $(y,0)$, has width 2λ and has direction $(2v,1)$. Therefore, if we fix $v \in \mathcal{V}$, a simple counting in the case of $\mathbb{R}$ shows that we must only consider values of y in the interval $[-4\pi\lambda^2 v - 3\lambda/2, 3\lambda/2]$, which has length $3\lambda + 4\pi\lambda^2 v$. Thus, since the spacial grid separation is λ, there are approximately $3 + 4\pi\lambda v$ values of y available. And since $|\mathcal{V}| \approx 2\lambda$, we count $4\pi\lambda \cdot 2\lambda = O(\lambda^2)$ tubes. In general, a similar reasoning in $\mathbb{R}^n$ gives $O(\lambda^{2n})$ tubes.

Hence we have selected only $\approx \lambda^{2n}$ tubes. Now for each time interval I_j we define

$$f_j = \sum_{T \cap \lambda^\epsilon q_j \neq \emptyset} f_T. \tag{3.36}$$

From the definition the support property of the statement arises. Indeed, it is consequence of the fact that it is a finite sum and that $\operatorname{supp} \widehat{f_T} \subset \operatorname{supp} \widehat{f} + O(\lambda^{-1})$.

Considering this selection, we split $e^{it\Delta} f$ in two parts, and using (3.35) we obtain

$$\begin{aligned} |e^{it\Delta} f(x)| &= \left|\sum_T e^{it\Delta} f_T(x)\right| \leq \left|\sum_{T\cap\lambda^\epsilon q_j \neq \emptyset} e^{it\Delta} f_T(x)\right| + \left|\sum_{T\cap\lambda^\epsilon q_j = \emptyset} e^{it\Delta} f_T(x)\right| \\ &\leq \left|e^{it\Delta} f_j(x)\right| + C_N \lambda^{-n/2} \|f\|_2 \sum_{T\cap\lambda^\epsilon q_j = \emptyset} \left(1 + \frac{|x-(y+4\pi t v)|}{\lambda}\right)^{-N}. \end{aligned} \tag{3.37}$$

The remaining tubes $T \cap \lambda^\epsilon q_j = \emptyset$ need to be managed. For $(x,t) \in q_j$ we know that

$$|x - y - 4\pi t v| = \operatorname{dist}(T,(x,t)) \geq \operatorname{dist}(T,q_j).$$

Now, let us fix the direction of the tubes, v. Since $\mathcal{Y} = \lambda\mathbb{Z}^n$, we can find a unique point $\lambda k \in \lambda\mathbb{Z}^n$ in each tube T. Moreover, since $T \cap \lambda^\epsilon q_j = \emptyset$, it is necessary that $|k|\lambda > \lambda^{1+\epsilon}$, thus implying $|k| > \lambda^\epsilon$. Also,

$$\operatorname{dist}(T,q_j) \approx |k|\lambda - \lambda \approx \lambda|k|,$$

because $|k| > \lambda^\epsilon$ implies $|k| - 1 \approx |k|$. Therefore, we have

$$\left(1 + \frac{|x-(y+4\pi t v)|}{\lambda}\right)^{-N} \leq (1+|k|)^{-N} \approx |k|^{-N}.$$

On the other hand, we have said before that there are $\approx \lambda^n$ different directions in $\mathcal{V}$, so the second part in (3.37) can be bounded by

$$C_N \lambda^{n/2} \|f\|_2 \sum_{k\in\mathbb{Z}^n, |k|>\lambda^\epsilon} |k|^{-N}.$$

This last sum can be managed by means of an integral, since

$$\sum_{k\in\mathbb{Z}^n, |k|>\lambda^\epsilon} |k|^{-N} \approx \int_{|x|>\lambda^\epsilon} \frac{dx}{|x|^N} = C_N(\lambda^\epsilon)^{n-N}$$

whenever $N > n$. Because of that, and thanks to this ϵ term we have introduced, and because of the freedom to choose N as big as we wish, we end up in (3.37) with

$$|e^{it\Delta} f(x)| \leq \left| e^{it\Delta} f_j(x) \right| + C_N \lambda^{-N} \|f\|_2,$$

which is the second property in the statement.

Now let us prove the first one. As we saw in (3.33), we can work with the norm of each wave separately, since

$$\|f_j\|_2^2 = \| \sum_{T\cap\lambda^\epsilon q_j \neq \emptyset} f_T \|_2^2 \approx \sum_{T\cap\lambda^\epsilon q_j \neq \emptyset} \|f_T\|_2^2,$$

so summing in j we get

$$\sum_{j=1}^{\lambda} \|f_j\|_2^2 \approx \sum_{j=1}^{\lambda} \sum_{T\cap\lambda^\epsilon q_j \neq \emptyset} \|f_T\|_2^2.$$

Observe that we are summing, for each j, the tubes that contribute in q_j. We want to change the order of summation. For that, we need to count how many levels $\lambda^\epsilon q_j$ each tube T has effect in. In other words, for each T we need to count the number of q_j it intersects. We know that the tube has direction $(2v, 1)$, so a space move of magnitude λ makes the tube grow $2|v|\lambda$. Now, since the tube has a base of length (diameter) λ, it has a vertical section of height $2|v|\lambda$. Also, since we are considering the levels are dilated by a λ^ϵ factor, the tube increases its height by $2|v|\lambda^{1+\epsilon}$. Therefore, the tube T occupies a height less than $4|v|\lambda^{1+\epsilon}$. Remember that each q_j has height λ, so the tube can intersect at most $4|v|\lambda^\epsilon$ levels q_j.

On the other hand, recall that $\operatorname{supp} \widehat{f} \subset A(1)$ and that $\widehat{f_T}$ has support in $B(v, \lambda^{-1/2})$. Also $\operatorname{supp} \widehat{f_T} \subset \operatorname{supp} f + O(\lambda^{-1/2}) = A(1) + O(\lambda^{-1/2})$. Since $\lambda \gg 1$, this error is very small and hence we could say that $\operatorname{supp} \widehat{f_T} \subset A(1/4, 2)$. This means that

$$\operatorname{supp} \widehat{f_T} \subset A(1/4, 2) \cap B(v, \lambda^{-1/2}),$$

so in order for it not to be empty, we must ask $|v| \approx 1$. In other words, it cannot be neither big nor small. Thus, each tube T intersects $\approx \lambda^\epsilon$ levels and the previous sum turns into

$$\sum_{j=1}^{\lambda} \sum_{T\cap\lambda^\epsilon q_j \neq \emptyset} \|f_T\|_2^2 \approx \sum_{T\in\mathcal{T}(\lambda)} \lambda^\epsilon \|f_T\|_2^2.$$

And for the norm equivalence wave packet decomposition properties seen in (3.33), we write

$$\sum_{j=1}^{\lambda} \sum_{T\cap\lambda^\epsilon q_j\neq\emptyset} \|f_T\|_2^2 \approx \lambda^\epsilon \sum_{T\in\mathcal{T}(\lambda)} \|f_T\|_2^2 \approx \lambda^\epsilon \|f\|_2^2.$$

This concludes the proof because if we rename $\epsilon \leftrightarrow 2\epsilon$ we have

$$\left(\sum_{j=1}^{\lambda} \|f_j\|_2^2\right)^{1/2} \approx \left(\sum_{j=1}^{\lambda} \sum_{T\cap\lambda^\epsilon q_j\neq\emptyset} \|f_T\|_2^2\right)^{1/2} \approx \lambda^\epsilon \|f\|_2.$$

□

Remark 3.12. Before the proof of Lemma 3.10 we have said that the proof of Lemma 3.11 is almost the same. Indeed, the proof follows the same steps, but now we have to consider the wave-packet decomposition at scale λ. For each partition cube Q_l, we concentrate the waves whose tubes intersect Q_l into a block to define

$$f_l = \sum_{T\cap\lambda^\epsilon(Q_l\times[0,\lambda])\neq\emptyset} f_T$$

and the rest of the proof follows by repeating the steps (from time to time having to fix some calculation because of the change $\lambda \leftrightarrow \lambda^2$ we have performed). For the inequality of norms, we have measured the occupation of each tube vertically in $Q(\lambda)$. Now, since the partition is in space and the time interval $[0,\lambda]$ is fixed we will have to measure the occupation of each tube horizontally up to time λ.

The next property we are going to state is about a time-rescaling property when considering mixed norms of the solution $e^{it\Delta}f$. It is, as we will see, a consequence of Lemma 3.10.

Lemma 3.13. *Let $q,r \geq 2$. Suppose that for functions f with* $\operatorname{supp}\widehat{f} \subset A(\lambda)$ *we have*

$$\|e^{it\Delta} f\|_{L_x^q L_t^r(Q(1)\times[0,\lambda^{-1}])} \leq C\lambda^\alpha \|f\|_2. \tag{3.38}$$

Then, for any $\epsilon > 0$ we have

$$\|e^{it\Delta} f\|_{L_x^q L_t^r(Q(1)\times[0,1])} \leq C_\epsilon \lambda^{\alpha+\epsilon} \|f\|_2. \tag{3.39}$$

Proof. We first rescale the problem. By the mixed norm, we mean, as usual,

$$\|e^{it\Delta} f\|_{L_x^q L_t^r(Q(1)\times[0,\lambda^{-1}])} = \left(\int_{Q(1)} \left(\int_0^{\lambda^{-1}} |e^{it\Delta} f(x)|^r\, dt\right)^{q/r} dx\right)^{1/q},$$

and by changing variables $x \to x/\lambda, t \to t/\lambda^2$ we see that (3.38) turns into

$$\frac{1}{\lambda^{2/r+n/q}} \|e^{it\Delta} f_{1/\lambda}\|_{L_x^q L_t^r(Q(\lambda)\times[0,\lambda])} \leq C\lambda^{\alpha-n/2} \|f_{1/\lambda}\|_2, \tag{3.40}$$

where $f_{1/\lambda}(x) = f(x/\lambda)$. Observe that if $\mathscr{F}(f_{1/\lambda})(\xi) = \lambda^n \widehat{f}(\lambda\xi)$, so if $\operatorname{supp}\widehat{f} \subset A(\lambda)$, then $\operatorname{supp}\mathscr{F}(f_{1/\lambda}) \subset A(1)$. By the same change of variables, (3.39) turns into

$$\frac{1}{\lambda^{2/r+n/q}} \|e^{it\Delta} f_{1/\lambda}\|_{L_x^q L_t^r(Q(\lambda)\times[0,\lambda^2])} \le C_\epsilon \lambda^{\alpha-n/2+\epsilon} \|f_{1/\lambda}\|_2. \tag{3.41}$$

Hence, it is enough to see that (3.38) implies (3.39) for functions with Fourier support in $A(1)$ and substituting the working region to $Q(\lambda)\times[0,\lambda]$ and $Q(\lambda)\times[0,\lambda^2]$ respectively.

For this, consider the decomposition given in Lemma 3.10, so partition $[0,\lambda^2]$ into disjoint intervals $I_j = [t_j, t_{j+1})$ of length $\approx \lambda$. Then, we have functions f_j satisfying the properties given in the lemma. Since we are going to work with this decomposition, we want to translate the hypothesis (3.40) to each of the f_j functions. We will use the fact that the Schrödinger operator generates a group

$$\{e^{it\Delta} \mid t \in \mathbb{R}\}, \qquad e^{it_1\Delta} e^{it_2\Delta} = e^{i(t_1+t_2)\Delta}, \quad \forall t_1, t_2 \in \mathbb{R}.$$

Then, since I_j has length $\approx \lambda$, we translate $s = t - t_j$ to obtain

$$\begin{aligned}
\|e^{it\Delta} f_j\|_{L_x^q L_t^r(Q(\lambda)\times I_j)} &= \left\| \left(\int_{I_j} |e^{it\Delta} f_j(x)|^r \, dt \right)^{1/r} \right\|_{L_x^q(Q(\lambda))} \\
&= \left\| \left(\int_{I_j} |e^{i(t-t_j)\Delta} e^{it_j\Delta} f_j(x)|^r \, dt \right)^{1/r} \right\|_{L_x^q(Q(\lambda))} \\
&= \left\| \left(\int_0^\lambda |e^{is\Delta} e^{it_j\Delta} f_j(x)|^r \, ds \right)^{1/r} \right\|_{L_x^q(Q(\lambda))} \\
&= \|e^{is\Delta} e^{it_j\Delta} f_j\|_{L_x^q L_t^r(Q(\lambda)\times[0,\lambda])}.
\end{aligned}$$

Observe that by what we saw in the formation of $e^{it\Delta} f$ in (0.5), we know that $\mathscr{F}(e^{it_j\Delta} f_j) = e^{-4\pi^2 it_j|\xi|^2} \widehat{f}_j$, so since $\widehat{f}_j$ has almost the same support as $\widehat{f}$, we can use the hypothesis and say that

$$\|e^{is\Delta} e^{it_j\Delta} f_j\|_{L_x^q L_t^r(Q(\lambda)\times[0,\lambda])} \le C\lambda^\alpha \|e^{it_j\Delta} f_j\|_{L^2} = C\lambda^\alpha \|f_j\|_{L^2},$$

so we obtain

$$\|e^{it\Delta} f_j\|_{L_x^q L_t^r(Q(\lambda)\times I_j)} \le C\lambda^\alpha \|f_j\|_{L^2}. \tag{3.42}$$

We now need to consider two cases. We first suppose that $q \ge r$. Then, since we want to take advantage of (3.42), we split the time integral into the subintervals I_j and we use the bounds in Lemma 3.10 to write

$$\begin{aligned}
\|e^{it\Delta} f\|^q_{L_x^q L_t^r(Q(\lambda)\times[0,\lambda^2])} &= \int_{Q(\lambda)} \left(\sum_{j=1}^{\lambda} \int_{I_j} |e^{it\Delta} f(x)|^r \, dt \right)^{q/r} dx \\
&\le \int_{Q(\lambda)} \left(\sum_{j=1}^{\lambda} \int_{I_j} (|e^{it\Delta} f_j(x)| + C\lambda^{-N} \|f\|_2)^r \, dt \right)^{q/r} dx.
\end{aligned}$$

We can take the power r inside by paying a constant 2^r and sum each term separately. Since the second sum is a sum of constants and does not depend on the integral parameter, we can write

$$\begin{aligned} &2^r \int_{Q(\lambda)} \left(\sum_{j=1}^{\lambda} \int_{I_j} |e^{it\Delta} f_j(x)|^r \, dt + \lambda^2 (C\lambda^{-N} \|f\|_2)^r \right)^{q/r} dx \\ &2^r 2^{q/r} \int_{Q(\lambda)} \left(\sum_{j=1}^{\lambda} \int_{I_j} |e^{it\Delta} f_j(x)|^r \, dt \right)^{q/r} dx + |Q(\lambda)| \lambda^{2q/r} (C\lambda^{-N} \|f\|_2)^q, \end{aligned} \tag{3.43}$$

where again we have taken the power q/r inside by paying a constant $2^{q/r}$. Observe that the right-hand side term is controlled by $\lambda^{-Nq+n+2q/r}$ and works for every $N \in \mathbb{N}$ so by making N as big as we wish we can make it as small as needed. Hence we focus on the left-hand side term. Indeed, if for simplicity we call $F_j(x) = \int_{I_j} |e^{it\Delta} f_j(x)|^r \, dt$, this term is, save constants,

$$\int_{Q(\lambda)} \left(\sum_{j=1}^{\lambda} F_j(x) \right)^{q/r} dx = \left\| \sum_{j=1}^{\lambda} F_j(x) \right\|_{L_x^{q/r}(Q(\lambda))}^{q/r},$$

and Minkowski's inequality is available because $q/r \geq 1$ and thus we are working with norms. Hence,

$$\begin{aligned} \left\| \sum_{j=1}^{\lambda} F_j(x) \right\|_{L_x^{q/r}(Q(\lambda))}^{q/r} &\leq \left(\sum_{j=1}^{\lambda} \left\| F_j(x) \right\|_{L_x^{q/r}(Q(\lambda))} \right)^{q/r} \\ &= \left(\sum_{j=1}^{\lambda} \left\| e^{it\Delta} f_j \right\|_{L_x^q L_t^r(Q(\lambda) \times I_j)}^r \right)^{q/r}. \end{aligned}$$

Therefore, we have been able to obtain

$$\|e^{it\Delta} f\|_{L_x^q L_t^r(Q(\lambda) \times [0,\lambda^2])}^q \leq C \left(\sum_{j=1}^{\lambda} \left\| e^{it\Delta} f_j \right\|_{L_x^q L_t^r(Q(\lambda) \times I_j)}^r \right)^{q/r} + C\lambda^{-N} \|f\|_2^q \tag{3.44}$$

By (3.44) and (3.42) we can write

$$\begin{aligned} \|e^{it\Delta} f\|_{L_x^q L_t^r(Q(\lambda) \times [0,\lambda^2])}^q &\leq C \left(\sum_{j=1}^{\lambda} C\lambda^{r\alpha} \|f_j\|_{L^2}^r \right)^{q/r} + C\lambda^{-N} \|f\|_2^q \\ &\leq C\lambda^{q\alpha} \left(\sum_{j=1}^{\lambda} \|f_j\|_{L^2}^2 \right)^{q/2} + C\lambda^{-N} \|f\|_2^q, \end{aligned} \tag{3.45}$$

where the last equality comes from the fact that p-norms in $\mathbb{R}^n$ are decreasing and $r \geq 2$. Also by the norm inequality in Lemma 3.10 we can carry on bounding to obtain

$$\begin{aligned} \|e^{it\Delta} f\|_{L_x^q L_t^r(Q(\lambda) \times [0,\lambda^2])}^q &\leq C\lambda^{q\alpha} C_\epsilon \lambda^{q\epsilon} \|f\|_2^q + C\lambda^{-N} \|f\|_2^q \\ &= C_\epsilon \left(\lambda^{q(\alpha+\epsilon)} + \lambda^{-N} \right) \|f\|_2^q \\ &\leq C_\epsilon \lambda^{q(\alpha+\epsilon)} \|f\|_2^q, \end{aligned}$$

which is precisely the result we sought.

Consider now $q < r$. In this case, Minkowski's inequality cannot be used since $q/r < 1$. Nevertheless, this condition will allow us to use another property we could not before. Indeed, we can take the same steps as in the case $q \geq r$ up to (3.43). Now observe that we have a power of a sum, but since the power is smaller than 1, we can take it inside without having to pay any constant depending on the number of summands. In other words, we can write

$$\begin{aligned}
&\|e^{it\Delta} f\|^q_{L^q_x L^r_t(Q(\lambda)\times[0,\lambda^2])} \\
&\quad \leq C \int_{Q(\lambda)} \sum_{j=1}^{\lambda} \left(\int_{I_j} |e^{it\Delta} f_j(x)|^r \, dt \right)^{q/r} dx + C\lambda^{-N} \|f\|_2^q \\
&\quad = C \sum_{j=1}^{\lambda} \|e^{it\Delta} f_j\|^q_{L^q_x L^r_t(Q(\lambda)\times I_j)} + C\lambda^{-N} \|f\|_2^q.
\end{aligned}$$

Again, by (3.42) we have

$$\begin{aligned}
\|e^{it\Delta} f\|^q_{L^q_x L^r_t(Q(\lambda)\times[0,\lambda^2])} &\leq C\lambda^{q\alpha} \sum_{j=1}^{\lambda} \|f_j\|^q_{L^2} + C\lambda^{-N} \|f\|_2^q \\
&\leq C\lambda^{q\alpha} \left(\sum_{j=1}^{\lambda} \|f_j\|^2_{L^2} \right)^{q/2} + C\lambda^{-N} \|f\|_2^q,
\end{aligned}$$

and we have obtained the same estimate as in (3.45), so repeating the steps of the previous case we obtain the full result and the lemma is proven. □

In the following lemma we present a maximal estimate. To denote the cube centred at a point ξ_0 and of side-length $\rho > 0$, we will write $Q(\xi_0, \rho)$.

Lemma 3.14. *Consider $\lambda \gg 1$ and a function $f \in \mathscr{S}(\mathbb{R}^n)$ whose Fourier transform is supported in $Q(\xi_0, \rho) \subset A(\lambda)$ with $\rho \geq 1$. Then,*

$$\left\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f| \right\|_{L^2(\mathbb{R}^n)} \leq C\rho^{1/2} \|f\|_{L^2}.$$

Proof. Observe that a translation $\xi \to \xi + \xi_0$ in the definition of $e^{it\Delta} f(x)$ implies

$$|e^{it\Delta} f(x)| = \left| \int_{\mathbb{R}^n} \widehat{f}(\xi + \xi_0) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2 - 4\pi t \xi\cdot\xi_0)} \, d\xi \right|,$$

where the phase terms not depending on ξ disappear for the effect of the absolute value. We want to bound the L^2-norm of the supremum of the above expression. Observe that we have Lemma 3.9 at hand , so by fixing $\mu = \lambda\rho$, we have

$$\sup_{0<t<\lambda^{-1}} |e^{it\Delta} f(x)| \leq C(I + II + III),$$

where

$$I = \left| e^{i0\Delta} f(x) \right|, \tag{3.46}$$

$$II = (\lambda\rho)^{1/2} \left\| e^{it\Delta} f(x) \right\|_{L_t^2(0,\lambda^{-1})}, \tag{3.47}$$

and for the derivative using the alternative expression we have written above,

$$III = (\lambda\rho)^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^n} \widehat{f}(\xi+\xi_0) e^{2\pi i (x\cdot\xi - 2\pi t(|\xi|^2 + 2\xi\cdot\xi_0))} (4\pi^2(|\xi|^2 + 2\xi\cdot\xi_0)) d\xi \right\|_{L_t^2(0,\lambda^{-1})}. \tag{3.48}$$

By the triangle inequality, we need to bound $\|I\|_2$, $\|II\|_2$ and $\|III\|_2$. First of all, we see that since $f \in \mathscr{S}$,

$$I = \left| e^{i0\Delta} f(x) \right| = |f(x)|,$$

so trivially $\|I\|_2 = \|f\|_2$. In the case of II, we see that

$$\|II\|_2^2 = \lambda\rho \int_{\mathbb{R}^n} \int_0^{\lambda^{-1}} |e^{it\Delta} f(x)|^2 \, dt \, dx,$$

and by Fubini's theorem we change the order to obtain

$$\|II\|_2^2 = \lambda\rho \int_0^{\lambda^{-1}} \|e^{it\Delta} f\|_2^2 \, dt = \lambda\rho \int_0^{\lambda^{-1}} \|f\|_2^2 \, dt = \rho \|f\|_2^2,$$

where we have used the fact that $e^{it\Delta}$ is an isometry in L^2. So let us focus on III. Observe that the integral in (3.48) is indeed an integral only on $B(0,\rho)$ because of the support condition of $\widehat{f}$. This fact implies a control over the extra term which has appeared by the effect of the derivative. Indeed,

$$||\xi|^2 + 2\xi\cdot\xi_0| \le |\xi|^2 + 2|\xi||\xi_0| \le \rho^2 + 2\rho\lambda$$

because $Q(\xi_0,\rho) \subset A(\lambda)$ implies $|\xi_0| \le \lambda$. Moreover, that same condition obliges $\rho < \lambda$, so we get $\rho^2 + 2\rho\lambda < \rho\lambda + 2\rho\lambda = 3\rho\lambda$. Hence, by this bound and then reversing the translation by $\xi \to \xi - \xi_0$, we get

$$\|III\|_2^2 \le \frac{(12\pi\lambda\rho)^2}{\lambda\rho} \int_{\mathbb{R}^n} \left\| e^{it\Delta} f(x) \right\|_{L_t^2(0,\lambda^{-1})}^2 dx.$$

Again by changing the order by Fubini and by the isometry property of the Scrödinger operator, we see that

$$\|III\|_2^2 \le C\lambda\rho \int_0^{\lambda^{-1}} \|f\|_2^2 \, dt = C\rho \, \|f\|_2^2.$$

Once the bounds needed obtained, we can write

$$\left\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f| \right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_2 (1 + \rho^{1/2}).$$

Observe that since $\lambda \gg 1$, we can expect $\rho \ge 1$, so $1 + \rho^{1/2} \le 2\rho^{1/2}$ and

$$\left\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f| \right\|_{L^2(\mathbb{R}^n)} \le C\rho^{1/2} \|f\|_2$$

as we wished. □

3.4 Proof of Theorem 3.1

In previous chapters we have been able to obtain a maximal estimate of the solution $e^{it\Delta}f$ in terms of the H^s norm of the datum f, like

$$\|\sup_t |e^{it\Delta}f|\|_{L^p} \leq C\|f\|_{H^s}$$

for some $p \in (1,\infty)$ and for $f \in \mathscr{S}(\mathbb{R}^n)$. The norm in the left-hand side is taken in some time interval including zero and in some space which could be a ball or the whole space. The result for regular functions could later be generalised to H^s functions. But observe that the properties we have analysed in Section 3.3 apply to functions whose Fourier support is included in an annulus. This is not the general situation, but we can manage to reduce the problem to annuli. Indeed, let us suppose that we have an estimate such as

$$\|\sup_{0<t<1} |e^{it\Delta}f|\|_{L^2(Q(2\pi))} \leq C\|f\|_{H^s} \tag{3.49}$$

for $f \in \mathscr{S}$ with $\operatorname{supp}\widehat{f} \subset A(\lambda)$ for some $\lambda \gg 1$. This 2π size factor might seem strange, but it is there by matters of technicalities regarding the choice of the definition for the Fourier transform. The situation can then be managed by means of a Littlewood-Paley-type decomposition. Consider $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ in $A(1)$ and with support in $A(1/4,2)$. Consider the dilations $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, which have support in $A(2^{j-2},2^{j+1})$. Then it is clear that the sum $\sum_{j\in\mathbb{Z}}\varphi_j(\xi)$ is always finite and does not vanish. We define

$$\psi(\xi) = \frac{\varphi(|\xi|)}{\sum_{j\in\mathbb{Z}}\varphi_j(|\xi|)}.$$

Then it is clear that $\sum_{k\in\mathbb{Z}}\psi(2^{-k}\xi) = 1$ and that $\operatorname{supp}\psi_j = A(2^{j-2},2^{j+1})$. Using this decomposition we split the Fourier transform of f defining

$$\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi), \quad \forall j \in \mathbb{Z}.$$

Clearly, $\operatorname{supp}\widehat{S_j f} \subset A(2^{j-2},2^{j+1})$, and

$$\sum_{j\in\mathbb{Z}}\widehat{S_j f}(\xi) = \widehat{f}(\xi)\sum_{j\in\mathbb{Z}}\psi(2^{-j}\xi) = \widehat{f}(\xi).$$

We only can work with annuli of big radius, so we will concentrate every small annuli. Indeed, for some $M \gg 1$ we can write

$$1 = \sum_{j\in\mathbb{Z}}\psi(2^j\xi) = \sum_{j\leq M}\psi(2^{-j}\xi) + \sum_{j>M}\psi(2^{-j}\xi),$$

where we call $\sum_{j\leq M}\psi(2^{-j}\xi) = \psi_M(\xi) = \chi_{B(0,2^{M+1})}(\xi)$. Then

$$\widehat{f}(\xi) = \widehat{S_M f}(\xi) + \sum_{j>M}\widehat{S_j f}(\xi),$$

where $\widehat{S_M f} = \psi_M \widehat{f}$ and $\operatorname{supp}\widehat{S_M f} \subset B(0,2^{M+1})$. We see that (3.49) is available for $\widehat{S_j f}$ with $j > M$, but we need to manage the situation of $\widehat{S_M f}$. We can use the same procedure as in the proof of Lemma 3.14. Indeed, if we use Lemma 3.9 with $p = 2$ and $\mu = 1$, and denoting $S_M f = g$ for simplicity, we have

$$\sup_{0<t<1} |e^{it\Delta} g(x)| \le C \left(|e^{i0\Delta} g(x)| + \|e^{it\Delta} g(x)\|_{L^2_t} + \left\| \frac{d}{dt} e^{it\Delta} g(x) \right\|_{L^2_t} \right).$$

We have to analyse the L^2-norms in space. Observe that since $g \in \mathscr{S}$, we have $|e^{i0\Delta} g(x)| = |g(x)|$, so the first summand is $\|g\|_2$. For the second one, by Fubini,

$$\| \|e^{it\Delta} g(x)\|_{L^2_t} \|_2^2 = \int_{Q(2\pi)} \int_0^1 |e^{it\Delta} g(x)|^2 \, dt \, dx = \int_0^1 \|e^{it\Delta} g\|_2^2 \, dt,$$

and by the isometry property of the Schrödinger operator, we obtain simply $\|g\|_2$. Finally, for the derivative term, we see that

$$\left| \frac{d}{dt} e^{it\Delta} g(x) \right| = \left| \int_{B(0,2^M)} \widehat{g}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t |\xi|^2)} 4\pi^2 |\xi|^2 \, d\xi \right|,$$

and by bounding $|\xi|^2 \le 4^M$, we are in the same situation as with the second summand times a constant $4^{M+1}\pi^2$, so the bound is $C_M \|g\|_2^2$. Writing everything together, we obtain

$$\| \sup_{0<t<1} |e^{it\Delta} g(x)| \|_{L^2(Q(2\pi))} \le C_M \|g\|_2.$$

This is the version of (3.49) for the ball-support case because $\|g\|_2 \le \|g\|_{H^s}$.

Now that we have the bound for every element of the decomposition, observe that

$$e^{it\Delta} f(x) = \int \sum_{j=M}^{\infty} \widehat{S_j f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t |\xi|^2)} \, d\xi = \sum_{j=M}^{\infty} \int \widehat{S_j f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t |\xi|^2)} \, d\xi = \sum_{j=M}^{\infty} e^{it\Delta} S_j f(x).$$

The change of the order of the sum and the integral is justified by Fubini because $f \in \mathscr{S}$. Then, by the triangle inequality and basic properties for the supremum, we see that

$$\sup_{0<t<1} |e^{it\Delta} f(x)| \le \sum_{j=M}^{\infty} \sup_{0<t<1} |e^{it\Delta} S_j f(x)|$$

and hence with the norms and (3.49) we get

$$\| \sup_{0<t<1} |e^{it\Delta} f(x)| \|_{L^2(Q(2\pi))} \le \sum_{j=M}^{\infty} \| \sup_{0<t<1} |e^{it\Delta} S_j f(x)| \|_{L^2(Q(2\pi))} \le C \sum_{j=M}^{\infty} \|S_j f\|_{H^s}.$$

Since $\widehat{S_j f}$ is supported in $A(2^{j-2}, 2^{j+1})$ for $j > M$, we see that

$$\|S_j f\|_{H^s}^2 = \int_{A(2^{j-2},2^{j+1})} |\widehat{S_j f}|^2 (1 + |\xi|^2)^s \, d\xi \approx 2^{2js} \int_{A(2^{j-2},2^{j+1})} |\widehat{S_j f}|^2 \, d\xi = 2^{2js} \|S_j f\|_2^2.$$

Also recall that the bound we have obtained for the ball-supported element was an L^2-norm. Then, since $M \gg 1$, we can write

$$\| \sup_{0<t<1} |e^{it\Delta} f(x)| \|_{L^2(Q(2\pi))} \le C \sum_{j=M}^{\infty} 2^{js} \|S_j f\|_2.$$

This procedure is valid for any $s \geq 0$. Now, if we fix $s' = s + \epsilon$ for $\epsilon > 0$, we see that $2^{js} = 2^{js'}2^{-j\epsilon}$, and since $2^{js'}\|S_j f\|_2 \approx \|S_j f\|_{H^{s'}}$, we obtain

$$\| \sup_{0<t<1} |e^{it\Delta} f(x)| \|_{L^2(Q(2\pi))} \leq C \sum_{j=M}^{\infty} 2^{-j\epsilon} \|S_j f\|_{H^{s'}}.$$

And what is more, since ψ is bounded,

$$\|S_j f\|^2_{H^{s'}} = \int_{\mathbb{R}^n} |\psi(2^{-j}\xi)|^2 |\widehat{f}(\xi)|^2 (1+|\xi|^2)^{s'} \, d\xi \leq C \|f\|^2_{H^{s'}}.$$

This allows us to have a geometric convergent sum $\sum 2^{-j\epsilon} = C_\epsilon$, so we see that for $s' = s + \epsilon$,

$$\| \sup_{0<t<1} |e^{it\Delta} f(x)| \|_{L^2(Q(2\pi))} \leq C_\epsilon \|f\|^2_{H^{s'}}. \tag{3.50}$$

This is the maximal estimate we were looking for. Observe that s is the exponent we had in the result for the annuli, (3.49). This shows that the annulus property can be generalised to every function for any exponent $s' > s$ but not for s itself. Therefore, the preceding calculations show that to prove Theorem 3.1 it is enough to prove the following theorem.

Theorem 3.15. *Let $\lambda \gg 1$ and $\epsilon > 0$. Then,*

$$\| \sup_{0<t<1} |e^{it\Delta} f| \|_{L^2(Q(2\pi))} \leq C \|f\|_{H^{3/8+\epsilon}}$$

for every $f \in \mathscr{S}$ such that $\widehat{f}$ is supported in $A(\lambda)$.

Here, $s = 3/8 + \epsilon$, and thus the estimate for a general function is $s' = 3/8 + \epsilon + \epsilon'$. Since both ϵ, ϵ' can be done as small as we wish, we will obtain the estimate for any $s' > 3/8$ as Theorem 3.1 asserts.

Remark 3.16. We know that an estimate like the one in (3.50) implies Theorem 3.1 in the same way we did in Chapter 1. But observe that for being able to apply that reasoning we need to cover the whole space. Here we have only obtained an estimate for a cube $Q(2\pi)$. Nevertheless, it is easy to see that $e^{it\Delta}$, as an operator, is translation invariant. With this we mean that $e^{it\Delta} f(x + x_0) = e^{it\Delta}(f(\cdot + x_0))(x)$. It is also a fact that translations do not change the H^s-norm. Hence, estimate (3.50) is also true for any cube of side-length 2π. Now we can cover the whole space with countably many cubes. Since in each cube convergence will follow almost everywhere, so will happen in the whole space, thus obtaining the result of Theorem 3.1.

Proof of Theorem 3.15. Observe that we have said that

$$\|f\|_{H^{3/8+\epsilon}} \approx \lambda^{3/8+\epsilon} \|f\|_2,$$

so it is enough to prove

$$\| \sup_{0<t<1} |e^{it\Delta} f| \|_{L^2(Q(2\pi))} \leq C \lambda^{3/8+\epsilon} \|f\|_2. \tag{3.51}$$

Now Lemma 3.13 gives us the way to rescaling the problem. Indeed, our situation is the same if we consider $r = \infty$ and $q = 2$, so it shows that it is enough to prove that for $\epsilon > 0$,

$$\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f| \|_{L^2(Q(2\pi))} \leq C\lambda^{3/8+\epsilon} \|f\|_2 \tag{3.52}$$

whenever $\widehat{f}$ is supported in $A(\lambda)$. Observe that the extra ϵ term in Lemma 3.13 can be absorbed by the ϵ term we already have.

The first tool we will need to use to prove (3.52) is the Whitney-type decomposition analysed in Section 3.2. For convenience, the decomposition we are going to consider will not be exactly dyadic cubes but with λ as an amplification constant. Hence, we will have $l(\tau_k^j) = \lambda 2^{-j}$. Another variation we will make is that we will not consider every level of the decomposition, which we will later specify. For the moment, we write

$$\begin{aligned}
e^{it\Delta} f(x)^2 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{f}(\xi)\widehat{f}(\eta) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)} e^{2\pi i (x\cdot\eta - 2\pi t|\eta|^2)}\, d\xi\, d\eta \\
&= \sum_{j\in\mathbb{Z}} \sum_{\tau_k^j \sim \tau_{k'}^j} \int_{\tau_k^j \times \tau_{k'}^j} \widehat{f}(\xi)\widehat{f}(\eta) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)} e^{2\pi i (x\cdot\eta - 2\pi t|\eta|^2)}\, d\xi\, d\eta \\
&= \sum_{j\in\mathbb{Z}} \sum_{\tau_k^j \sim \tau_{k'}^j} \left(\int_{\tau_k^j} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi \right) \left(\int_{\tau_{k'}^j} \widehat{f}(\eta) e^{2\pi i (x\cdot\eta - 2\pi t|\eta|^2)}\, d\xi\, d\eta \right) \\
&= \sum_{j\in\mathbb{Z}} \sum_{\tau_k^j \sim \tau_{k'}^j} \left(e^{it\Delta} f_k^j(x) \right) \left(e^{it\Delta} f_{k'}^j(x) \right),
\end{aligned}$$

where $\widehat{f_k^j}(\xi) = \widehat{f}(\xi)\chi_{\tau_k^j}(\xi)$ is the restriction to the cube τ_k^j. Now if we write the L^2-norm formally as a L^1-norm and since $\sup|f|^2 = (\sup|f|)^2$ we can write

$$\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f| \|_{L^2(Q(2\pi))} = \| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f|^2 \|_{L^1(Q(2\pi))}^{1/2}.$$

Hence, for this and for Whitney's decomposition, to prove (3.52) is equivalent to proving

$$\| \sup_{0<t<\lambda^{-1}} \sum_{j\in\mathbb{Z}} \sum_{\tau_k^j \sim \tau_{k'}^j} |e^{it\Delta} f_k^j| |e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))} \leq C\lambda^{3/4+\epsilon} \|f\|_2^2. \tag{3.53}$$

Now, observe that it makes no sense to consider cubes τ_k^j which are too big. Indeed, since we are decomposing the Fourier space and $\operatorname{supp}\widehat{f} \subset A(\lambda)$, we do not need to consider cubes which have length greater than λ. We will also discard too small cubes. Indeed, we will ask $2^{-j} \geq \lambda^{-1/4}$, which is equivalent to $l(\tau_k^j) = \lambda 2^{-j} \geq \lambda^{3/4}$. Recall that the closer we are to the main diagonal, the smaller cubes we have. Since we are forcing not to have very small cubes, we will complete their place with cubes of the smallest possible size. Therefore, the decomposition we are considering is indeed

$$|e^{it\Delta} f(x)|^2 \leq \sum_{1\leq 2^j \leq \lambda^{1/4}} \sum_{\tau_k^j \sim \tau_{k'}^j} |e^{it\Delta} f_k^j(x)| |e^{it\Delta} f_{k'}^j(x)|,$$

where if close to the diagonal, the relation between cubes is not that one by Whitney, but we simply consider the cubes of length $\lambda^{3/4}$ that fill the gaps. In these last cases, since the product cubes are bigger than what they should be, the cubes τ_k^j and $\tau_{k'}^j$ have no separation even if we write $\tau_k^j \sim \tau_{k'}^j$.

Let us come back to (3.53). The supremum of the sum is smaller that the sum of the suprema if we consider positive functions, so we see that

$$\begin{aligned} \| \sup_{0<t<\lambda^{-1}} \sum_{1\leq 2^j \leq \lambda^{1/4}} \sum_{\tau_k^j \sim \tau_{k'}^j} |e^{it\Delta} f_k^j| |e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))} \\ \leq \sum_{1\leq 2^j \leq \lambda^{1/4}} \sum_{\tau_k^j \sim \tau_{k'}^j} \| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j| |e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))}, \end{aligned} \tag{3.54}$$

so we will try to obtain the bound for the inner norm. Fix j. Then, observe that for each k there are a finite number of k' related to it. Indeed, in $\mathbb{R}^2$, since a cube has 8 surrounding cubes, then we have to count the sons of all the surrounding cubes of the father of τ_k^j. Since each cube has 4 sons, we see that τ_k^j has at most 32 related cubes $\tau_{k'}^j$. In the case of cubes of length $\lambda^{3/4}$ for which the relation has been redefined, since adjacency is now allowed, we have to take the father cube into account too, so there will be at most 36 related cubes. In any case, it is a finite quantity. Based on this, if we were able to obtain

$$\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j| |e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))} \leq C\lambda^{3/4+\epsilon} \|f_k^j\|_2 \|f_{k'}^j\|_2, \tag{3.55}$$

then summing in the couples $\tau_k^j \sim \tau_{k'}^j$ for fixed j by Cauchy-Schwarz we obtain

$$C\lambda^{3/4+\epsilon} \sum_{k\sim k'} \|f_k^j\|_2 \|f_{k'}^j\|_2 \leq C\lambda^{3/4+\epsilon} \left(\sum_{k\sim k'} \|f_k^j\|_2^2 \right)^{1/2} \left(\sum_{k\sim k'} \|f_{k'}^j\|_2^2 \right)^{1/2}.$$

Observe that the symmetry of the relation makes the two sums be identical, and we see that

$$\sum_{k\sim k'} \|f_k^j\|_2^2 = \sum_k \sum_{k'\sim k} \|f_k^j\|_2^2 \leq 36 \sum_k \|f_k^j\|_2^2.$$

Recall that $\{\tau_k^j\}_k$ is a partition of $\mathbb{R}^2$, so

$$\sum_k \|f_k^j\|_2^2 = \sum_k \int_{\tau_k^j} |\widehat{f}(\xi)|^2 \, d\xi = \|\widehat{f}\|_2^2 = \|f\|_2^2.$$

Applying this in (3.54), we are left with

$$\sum_{1\leq 2^j \leq \lambda^{1/4}} C\lambda^{3/4+\epsilon} \|f\|_2^2 = C(\log_2 \lambda^{1/4}) \lambda^{3/4+\epsilon} \|f\|_2^2.$$

We can say now that since $\epsilon \log_2 \lambda = \log_2 \lambda^\epsilon \leq \lambda^\epsilon$, we have $\log_2 \lambda \leq \epsilon^{-1} \lambda^\epsilon$ and

$$\frac{C}{4} (\log_2 \lambda) \lambda^{3/4+\epsilon} \|f\|_2^2 \leq \frac{C}{4\epsilon} \lambda^{3/4+2\epsilon} \|f\|_2^2$$

which is precisely what we asked in (3.53). The constant depends on ϵ, but observe that the choice of ϵ determines the choice of the Sobolev space $H^{3/8+\epsilon}$, so each time it will be fixed. Hence we have shown that it is enough to prove (3.55) for every couple of related Whitney cubes. Let us prove it.

First of all, we consider the case of the cubes of length $\lambda^{3/4}$. This is the case when $2^{-j} = \lambda^{-1/4}$. Recall that these are the cubes which deal with the region close to the main diagonal. In this situation we use Lemma 3.14. Indeed, recalling $\widehat{f_k^j}(\xi) = \widehat{f}(\xi)\chi_{\tau_k^j}(\xi)$, we see that

$$\operatorname{supp}\widehat{f_k^j} = \operatorname{supp}\widehat{f} \cap \tau_k^j \subset A(\lambda) \cap \tau_k^j.$$

Hence we have a function whose Fourier transform is supported in a cube inside an annulus, the cube having side-length $\lambda^{3/4}$. Then Lemma 3.14 applies and

$$\Big\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j| \Big\|_{L^2(\mathbb{R}^n)} \le C\lambda^{3/8} \|f_k^j\|_{L^2}.$$

The same estimate we obtain for $f_{k'}^j$. The result now follows by basic properties for the supremum and Hölder's inequality. More precisely, we write

$$\begin{aligned} &\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j||e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))} \\ &\qquad \le \| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_{k'}^j| \|_{L^1(Q(2\pi))} \\ &\qquad \le \| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_k^j| \|_{L^2(Q(2\pi))} \| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f_{k'}^j| \|_{L^2(Q(2\pi))} \end{aligned}$$

which is by the recently remarked consequence of Lemma 3.14 bounded by the desired factor $\lambda^{3/4}\|f_k^j\|_2\|f_{k'}^j\|_2$. Observe that here we do not obtain the extra ϵ term in the exponent. This suggests that this is the good case and that the remaining case will be the one forcing the little gap.

We thus now focus on $2^j < \lambda^{1/4}$. The sizes of the cubes are hence greater than $\lambda^{3/4}$. We claim that we can rely on the following proposition.

Proposition 3.17. *Let $2^j < \lambda^{1/4}$ and consider any functions f, g such that their Fourier supports are in $B(\lambda\xi_0, 2^{-j}\lambda) \subset A(\lambda)$ for some $\xi_0 \in \mathbb{R}^2$. Also we suppose $d(\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g}) \approx 2^{-j}\lambda$. Then, for $\epsilon > 0$,*

$$\| \sup_{0<t<\lambda^{-1}} |e^{it\Delta} f e^{it\Delta} g| \|_{L^1(Q(2\pi))} \le C 2^{-j/2} \lambda^{3/4+\epsilon} \|f\|_2 \|g\|_2, \tag{3.56}$$

where the constant $C > 0$ may depend on ϵ but neither on the functions nor on λ or j.

A few remarks are necessary to clarify the convenience of this proposition. Observe that in our objective (3.55) the bound we give must be independent of j. The result of Proposition 3.17 shows a dependence on j that is trivially removable since $2^{-j/2} < 1$. On the other hand, recall that our functions f_k^j and $f_{k'}^j$ are supported in τ_k^j and $\tau_{k'}^j$ respectively, being cubes of side-length $2^{-j}\lambda$ and with a separation of $2^{-j}\lambda$. Also both functions are supported in the annuli $A(\lambda)$. The subtleties may lie in that the supports are cubes not balls, and if they are cubes close to the boundary of the annulus, they might even be truncated cubes. In any case, they will always be sets of diameter at most $\sqrt{2}2^{-j}\lambda$. We will simply work with balls for simplicity. Hence clearly functions f_k^j and $f_{k'}^j$ satisfy the hypotheses of Proposition 3.17 and we therefore obtain what we asked in (3.55). In what remains we will thus focus on proving Proposition 3.17.

To avoid confusion, in the following few lines we explain the way we will follow in this proof, which is rather long. The main idea is that we can obtain an estimate similar to (3.56) but with some exponent λ^α which is probably not as good as $3/4+\epsilon$. This is, we will obtain

$$\|\sup_{0<t<\lambda^{-1}} |e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi))} \le C2^{-j/2}\lambda^\alpha \|f\|_2\|g\|_2, \tag{3.57}$$

for some α. This will be done by using elementary calculations. Even if this first exponent is not at all precise, we will see that an iteration process will allow us to reach $3/4+\epsilon$.

So let us obtain this initial exponent. If in $|e^{it\Delta}f(x)|$ we take the absolute value inside the integral, we immediately realize that it is bounded by $\|\widehat{f}\|_{L^1}$ for all values of time. Therefore,

$$\|\sup_{0<t<\lambda^{-1}} |e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi))} \le |Q(2\pi)|\|\widehat{f}\|_1\|\widehat{g}\|_1 = C\|\widehat{f}\|_1\|\widehat{g}\|_1.$$

Now, by the support condition, Hölder's inequality implies

$$\|\widehat{f}\|_1 \le \|\widehat{f}\|_2|B(\lambda\xi_0, 2^{-j}\lambda)|^{1/2} = C2^{-j}\lambda\|\widehat{f}\|_2, \tag{3.58}$$

and Plancherel's identity makes the supremum be bounded by $C2^{-2j}\lambda^2\|f\|_2\|g\|_2$. We have obtained an exponent $\alpha = 2$.

Once we have the desired initial estimation, we will rescale the problem with several changes of variables. Indeed, in (3.57) we can dilate $x \to \lambda^{-1}x$ and $t \to \lambda^{-2}t$ so that

$$\|\sup_{0<t<\lambda^{-1}} |e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi))} = \frac{1}{\lambda^2}\|\sup_{0<t<\lambda} |e^{i\frac{t}{\lambda^2}\Delta}f\left(\frac{x}{\lambda}\right)e^{i\frac{t}{\lambda}\Delta}g\left(\frac{x}{\lambda}\right)|\|_{L^1(Q(2\pi\lambda))}.$$

Writing the definition of the Schrödinger operator, one can easily check that

$$e^{i\frac{t}{\lambda^2}\Delta}f\left(\frac{x}{\lambda}\right) = e^{it\Delta}(f_{1/\lambda})(x),$$

where $f_{1/\lambda}(x) = f(x/\lambda)$. Also $\|f_{1/\lambda}\|_2 = \lambda\|f\|_2$, so we can write (3.57) equivalently as

$$\|\sup_{0<t<\lambda} |e^{it\Delta}(f_{1/\lambda})e^{it\Delta}(g_{1/\lambda})|\|_{L^1(Q(2\pi\lambda))} \le C2^{-j/2}\lambda^\alpha\|f_{1/\lambda}\|_2\|g_{1/\lambda}\|_2.$$

Since $\widehat{f_{1/\lambda}}(\xi) = \lambda^2\widehat{f}(\lambda\xi)$, we see that the supports have been shrank, because $\operatorname{supp} f_{1/\lambda} = \lambda^{-1}\operatorname{supp}\widehat{f} \subset B(\xi_0, 2^{-j}) \subset A(1)$. Also the distance between the supports is now 2^{-j} instead of $\lambda 2^{-j}$. Thus, if we rename $f_{1/\lambda}$ by simply f, the information we have is that

$$\|\sup_{0<t<\lambda} |e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi\lambda))} \le C2^{-j/2}\lambda^\alpha\|f\|_2\|g\|_2, \tag{3.59}$$

for functions f,g with Fourier support in $B(\xi_0,2^{-j})\subset A(1)$ and $d(\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g})\approx 2^{-j}$.

Next we want to rescale the support so that we are able to work in a ball centred in the origin and with unity radius, for the sake of simplicity in future calculations. For that we first rotate the plane so that the ball is centred in e_1 instead of ξ_0. This is, we want to transform $\xi_0\to e_1$, where $e_1=(1,0)\in\mathbb{R}^2$. Assume we need a θ-rotation for that. Let R_θ be the corresponding rotation matrix. The symmetry properties of the usual sine and cosine functions show that $R_\theta^T=R_{-\theta}$, and because of that we can see that

$$\widehat{f\circ Rot_\theta}(\xi)=\int_{\mathbb{R}^2}f(Rot_\theta(x))e^{-2\pi i\xi\cdot x}\,dx=\int_{\mathbb{R}^2}f(x)e^{-2\pi i\xi\cdot Rot_{-\theta}(x)}\,dx=\widehat{f}(Rot_\theta(\xi)),$$

where we have changed of variables $x\to Rot_{-\theta}(x)$ and used the fact that

$$\xi\cdot Rot_{-\theta}(x)=\xi^TR_{-\theta}x=\xi^TR_\theta^Tx=(R_\theta\xi)^Tx=Rot_\theta(\xi)\cdot x.$$

Hence, if we call $f_\theta=f\circ Rot_\theta$, we see that

$$e^{it\Delta}f(x)=\int_{\mathbb{R}^2}\widehat{f}(Rot_\theta(\xi))e^{2\pi i(x\cdot Rot_\theta(\xi)-2\pi t|\xi|^2)}\,d\xi=e^{it\Delta}f_\theta(Rot_{-\theta}(x)).$$

Also observe that rotations do not affect the L^2-norm of a function, so $\|f_\theta\|_2=\|f\|_2$. Thus (3.59) can be equivalently written as

$$\|\sup_{0<t<\lambda}|e^{it\Delta}f_\theta(Rot_{-\theta}(x))e^{it\Delta}g_\theta(Rot_{-\theta}(x))|\|_{L^1(Q(2\pi\lambda))}\le C2^{-j/2}\lambda^\alpha\|f_\theta\|_2\|g_\theta\|_2,$$

where $\operatorname{supp}\widehat{f_\theta}=Rot_\theta(\operatorname{supp}\widehat{f})=B(|\xi_0|e_1,2^{-j})$. The rotation does not change the distance between the supports of $\widehat{f}$ and $\widehat{g}$. Here we have to make a little remark. Indeed, $Q(2\pi\lambda)$ is not rotation invariant. Nevertheless, in the beginning we could have started with a ball $B(2\pi)$ instead of $Q(2\pi)$, and the space invariance would be a fact here. But the point is that we will need to work with cubes. Thus, since $Q(2\pi\lambda)\subset B(2\pi\lambda)$, we would see that (3.59) implies

$$\|\sup_{0<t<\lambda}|e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi\lambda))}\le C2^{-j/2}\lambda^\alpha\|f\|_2\|g\|_2, \tag{3.60}$$

for f ad g Fourier supported in $B(|\xi_0|e_1,2^{-j})$ and with $d(\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g})\approx 2^{-j}$. By a dilation $\xi\to|\xi_0|\xi$ in the Fourier side we can have the support in $B(e_1,2^{-j})$. More precisely, (3.60) turns into

$$\|\sup_{0<t<\lambda}|e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(|\xi_0|2\pi\lambda))}\le C2^{-j/2}\lambda^\alpha\|f\|_2\|g\|_2$$

for $\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g}\subset B(e_1,2^{-j}/|\xi_0|)$ and $d(\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g})\approx 2^{-j}/|\xi_0|$. But since the hypotheses force $\lambda\xi_0\in A(\lambda)$, we see that $|\xi_0|\in(1/2,1)$ and we can work with $\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g}\subset B(e_1,2^{-j+1})$ and $d(\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g})\approx 2^{-j}$.

Once the supports have been sent to the horizontal axis, we translate to the origin by $\xi\to\xi+e_1$ and dilate by $\xi\to 2^{-j}\xi$. We know that translation in the Fourier side generates modulations, but they will be harmless when working with the absolute value. Indeed, what we get after the translation is

$$|e^{it\Delta}f(x)| = \left|\int_{B(0,2^{-j})} \widehat{f}(\xi+e_1)e^{2\pi i(x\cdot\xi-4\pi t\xi\cdot e_1-2\pi t|\xi|^2)}\,d\xi\right|,$$

and then the dilation gives us

$$2^{-2j}\left|\int_{B(0,1} \widehat{f}(2^{-j}\xi+e_1)e^{2\pi i(2^{-j}\xi\cdot(x-4\pi te_1)-2\pi t2^{-2j}|\xi|^2)}\,d\xi\right| = e^{i2^{-2j}t\Delta}(f_{j,e_1})(2^{-j}x-2^{-j+1}2\pi te_1).$$

Here we write f_{j,e_1} to denote $\widehat{f_{j,e_1}} = \widehat{f}(2^{-j}\xi+e_1)$. Observe that now these functions have support in $B(0,1)$ and that their supports have separation ≈ 1. If we come back to the space-time as in (3.60), after making $x \to 2^j x$ and $t \to 2^{2j}t$ we get

$$\begin{aligned}\|\sup_{0<t<\lambda}|e^{it\Delta}fe^{it\Delta}g|\|_{L^1(Q(2\pi\lambda))} \\ = 2^{-2j}\|\sup_{0<t<2^{-2j}\lambda}|e^{it\Delta}f_{j,e_1}(x-2^{j+1}2\pi te_1)e^{it\Delta}g_{j,e_1}(x-2^{j+1}2\pi te_1)|\|_{L^1(Q(2^{-j}2\pi\lambda))}.\end{aligned}$$

The norm of the new function f_{j,e_1} will only generate a 2^{-j} term with respect to that of f, by the effect of the dilation. More precisely, we get $\|f\|_2 = 2^{-j}\|f_{j,e_1}\|_2$. Then, again renaming these new functions as simply f and g, we see that (3.60) is equivalent to

$$\|\sup_{0<t<2^{-2j}\lambda}|e^{it\Delta}f(x-2^{j+1}2\pi te_1)e^{it\Delta}g(x-2^{j+1}2\pi te_1)|\|_{L^1(Q(2^{-j}2\pi\lambda))} \leq C2^{-j/2}\lambda^{\alpha}\|f\|_2\|g\|_2 \tag{3.61}$$

for functions f and g with Fourier support in $B(0,1)$ and such that $d(\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g}) \approx 1$. So we have reached the setting we were looking for.

In this point we claim that from (3.61) we can get

$$\begin{aligned}\|\sup_{0<t<2^{-2j}\lambda}|e^{it\Delta}f(x-2^{j+1}2\pi te_1)e^{it\Delta}g(x-2^{j+1}2\pi te_1)|\|_{L^1(Q(2^{-j}2\pi\lambda))} \\ \leq C2^{-j/2}\lambda^{\epsilon}(\lambda^{\alpha(1-\delta)}+\lambda^{3/4+c\delta})\|f\|_2\|g\|_2\end{aligned} \tag{3.62}$$

for any $0<\epsilon,\delta \ll 1$, where $c>0$ is a constant which does not depend on ϵ, δ nor α. If this property were true, we could start an iteration process to reach the desired exponent 3/4. We here show this argument. To avoid confusion, let $\gamma>0$ be the exponent of the statement of Proposition 3.17 so that we look for $\lambda^{3/4+\gamma}$. This γ is from now on fixed. Since α has been obtained before, we choose

$$\delta = \frac{\gamma}{c+\alpha/2}, \qquad \epsilon = \alpha\delta/2.$$

This way, the exponent of the second term in (3.62) is $3/4+c\delta+\epsilon = 3/4+\delta(c+\alpha/2) = 3/4+\gamma$. That is our first choice of δ,ϵ. Now we have two options regarding the two exponents.

- If $\alpha(1-\delta) \leq 3/4+c\delta$, then since $\lambda \gg 1$ we can say that

  $$\lambda^{\epsilon}(\lambda^{\alpha(1-\delta)}+\lambda^{3/4+c\delta}) \leq 2\lambda^{\epsilon}\lambda^{3/4+c\delta} = 2\lambda^{3/4+\gamma}$$

  and we are done.

- If $\alpha(1-\delta) > 3/4 + c\delta$, then the bound we get is $2\lambda^{\epsilon}\lambda^{\alpha(1-\delta)}$. But the choice of ϵ makes it $2\lambda^{\alpha(1-\delta/2)}$. Call $\alpha' = \alpha(1-\delta/2) < \alpha$. The exponent has been improved, but since we have not yet reached 3/4, we use the claim again, where we keep δ and we choose

$$\delta' = \delta, \qquad \epsilon' = \frac{\alpha'\delta}{2} < \epsilon.$$

Hence, we have $\lambda^{\epsilon'}(\lambda^{\alpha'(1-\delta)} + \lambda^{3/4+c\delta})$, where the second exponent has not changed. Hence, we have again two options. If α' has improved enough, this is to say, if $\alpha'(1-\delta) \leq 3/4 + c\delta$, then the governing exponent is $3/4 + c\delta + \epsilon' < 3/4 + c\delta + \epsilon = 3/4 + \gamma$ and we have the result. The other possibility is that the improvement on the exponent is not enough, and the governing term is $\lambda^{\alpha'(1-\delta)+\epsilon'}$. The choice of ϵ makes it $\lambda^{\alpha'(1-\delta/2)}$, so we call the new exponent $\alpha'' = \alpha'(1-\delta/2) = \alpha(1-\delta/2)^2$ and we again can use the claim.

The iteration of this process shows that every time we obtain new exponents $\alpha^{(k)} = \alpha(1-\delta/2)^k$ and $\epsilon^{(k)} = \alpha^{(k)}\delta/2$. The exponent $\alpha^{(k)}$ tends to zero when $k \to \infty$, so there exists $k_0 \in \mathbb{N}$ so that $\alpha^{(k_0)} \leq 3/4 + c\delta$. In this case, the bound from the claim is

$$2^{k_0}\lambda^{\epsilon^{(k_0)}+3/4+c\delta} < 2^{k_0}\lambda^{\epsilon+3/4+c\delta} = 2^{k_0}\lambda^{3/4+\gamma}.$$

This shows that the exponent α in (3.61) can be substituted by $3/4+\gamma$. But to reach the statement of the proposition, or in other words, to unmake every rescaling we have done, we have to deal with the issue of the balls and cubes. Nevertheless, since we can take a ball included in $Q(2^{-j}2\pi\lambda)$, say $B(2^{-j}\pi\lambda)$, we will obtain the estimate of (3.61) for that ball too. If we are integrating on a ball, (3.61) is completely equivalent to the statement of the proposition. This implies that the initial $\alpha = 2$ we obtained for the result we sought can be improved until we reach $3/4+\gamma$. Hence the result of Proposition 3.17 follows with the norm taken in a ball instead of the cube prescribed. This conveys that we should focus on proving the claim (3.62).

Our setting is thus $(x,t) \in Q(2^{-j}2\pi\lambda) \times [0, 2^{-2j}\lambda] = Q$. Observe that since $1 \leq 2^j$, this is a rectangle whose width is bigger than its height. We see that the functions we have to deal with in (3.62) are translated by a factor depending on the time. In fact, if we consider some cube, the mapping $(x,t) \to (x - 2^{j+1}2\pi t e_1, t)$ produces an inclination of the tube in the direction of the first space coordinate. To be aware of this fact is important for the type of reasoning we are going to perform.

What we first do is to consider a slightly wider rectangle,

$$\overline{Q} = Q(5\cdot 2^{-j}2\pi\lambda) \times [0, 2^{-2j}\lambda],$$

and we divide it into cubes of side-length $2^{-2j}\lambda$. In other words, we are partitioning the domain in space, and the time-height is kept. Dividing the space area of $Q(5\cdot 2^{-j}2\pi\lambda)$ with the size of the partitioning cubes, we deduce that there will be $\approx 2^{2j}$ cubes of length $2^{-2j}\lambda$. This partition will be denoted by $\{\widetilde{B}_k\}_k$. From these cubes and inspired by the points in which we are evaluating the functions in (3.62), for each $\widetilde{B}_k$ we define

$$B_k = \{(x,t) : (x - 2^{j+1}2\pi t e_1, t) \in \widetilde{B}_k\} = \{(x + 2^{j+1}2\pi t e_1, t) : (x,t) \in \widetilde{B}_k\}.$$

The sets B_k are no longer cubes, but it is easy to see that they are parallelepipeds. As said, the linear change of variables only inclines the cubes $\widetilde{B}_k$ in the direction of e_1. Moreover, these inclined cubes have slope $\approx 2^{-j}$. The reason for we partition a wider rectangle $\overline{Q}$ instead of Q will become clear now. For when we incline $\overline{Q}$ as we have done with the cubes $\widetilde{B}_k$, the resulting parallelepiped covers the whole original rectangle Q. Since $\widetilde{B}_k$ are a partition of $\overline{Q}$, the union of B_k will be precisely the inclined version of $\overline{Q}$, so the covering property can be expressed as

$$Q \subset \bigcup_k B_k.$$

If we write the supremum term as an L^∞-norm, the expression we are intending to bound in (3.62) can also be written as

$$\|e^{it\Delta} f(x - 2^{j+1}2\pi t e_1) e^{it\Delta} g(x - 2^{j+1}2\pi t e_1)\|_{L^1_x L^\infty_t(Q)},$$

and for the covering property we have just mentioned, this norm is bounded by

$$\sum_k \|e^{it\Delta} f(x - 2^{j+1}2\pi t e_1) e^{it\Delta} g(x - 2^{j+1}2\pi t e_1)\|_{L^1_x L^\infty_t(B_k)} = \sum_k \|e^{it\Delta} f e^{it\Delta} g\|_{L^1_x L^\infty_t(\widetilde{B}_k)}. \tag{3.63}$$

The last equality comes from the definition of B_k and a translation in space.

We now want to use the result we saw in Lemma 3.11. Observe that in our case, the time goes as far as $2^{-2j}\lambda$, so we will have to make a substitution $\lambda \leftrightarrow 2^{-2j}\lambda$ according to the notation of the lemma. Indeed that is also the size of the partitioning cubes. Moreover, the Fourier support of our functions f and g is $B(1)$, so the lemma can be applied to obtain functions $\{f_k\}, \{g_k\}$ such that their Fourier supports are contained in $\operatorname{supp}\widehat{f} + O((2^{-2j}\lambda)^{-1/2})$. Moreover, we have the following estimates,

$$\left(\sum_k \|f_k\|_2^2\right)^{1/2} \le C_\epsilon (2^{-2j}\lambda)^\epsilon \|f\|_2, \qquad \left(\sum_k \|g_k\|_2^2\right)^{1/2} \le C_\epsilon (2^{-2j}\lambda)^\epsilon \|g\|_2 \tag{3.64}$$

and

$$|e^{it\Delta} f(x)| \le |e^{it\Delta} f_k(x)| + C(2^{-2j}\lambda)^{-N} \|f\|_2, \qquad |e^{it\Delta} g(x)| \le |e^{it\Delta} g_k(x)| + C(2^{-2j}\lambda)^{-N} \|g\|_2, \tag{3.65}$$

$\forall N \in \mathbb{N}$ whenever $(x,t) \in \widetilde{B}_k \times (0, 2^{-2j}\lambda)$. The question is if we can focus in each of the f_k and g_k blocks when analysing (3.63), and inequalities (3.65) suggest we can. On the one hand, by following the same steps as in (3.58), since $\operatorname{supp}\widehat{f} \subset B(1)$, we can say that

$$|e^{it\Delta} f(x)| \le C\|f\|_2.$$

On the other hand, we need to recall some properties of the proof of Lemma 3.11. From inequality (3.35) (observing that in Lemma 3.11 the scale was λ and not λ^2 as in the proof of Lemma 3.10) we have

$$|e^{it\Delta} f_T(x)| \leq C\lambda^{-n/4}\|f\|_2.$$

Also since each block f_k is defined as a finite sum (there were $O(\lambda^n)$ tubes to be considered) of wave-packets, we can write

$$|e^{it\Delta} f_k(x)| \leq \sum_{O(\lambda^n)} |e^{it\Delta} f_T(x)| \leq C\lambda^n \lambda^{-n/4}\|f\|_2$$

Hence, multiplying both inequalities in (3.65) and using these last two auxiliary inequalities we can write

$$|e^{it\Delta} f(x) e^{it\Delta} g(x)| \leq |e^{it\Delta} f_k(x) e^{it\Delta} g_k(x)| + C(2^{-2j}\lambda)^{-N}\|f\|_2\|g\|_2$$

for every $N \in \mathbb{N}$ and when $(x,t) \in \widetilde{B}_k \times (0, 2^{-2j}\lambda)$. Plug this in (3.63) to see that we want to bound

$$\sum_k \|e^{it\Delta} f_k e^{it\Delta} g_k\|_{L^1_x L^\infty_t(\widetilde{B}_k)} + C\sum_k (2^{-2j}\lambda)^{-N}\|f\|_2\|g\|_2\|\chi_{\widetilde{B}_k}\|_{L^1_x L^\infty_t(\widetilde{B}_k)}. \tag{3.66}$$

We know that there are approximately 2^{2j} cubes $\widetilde{B}_k$, and each has measure $(2^{-2j}\lambda)^2$. Thus, the second summand is bounded by

$$C(2^{-2j}\lambda)^{-N+2} 2^{2j}\|f\|_2\|g\|_2 \leq C(2^{-2j}\lambda)^{-N+2}\lambda^{1/2}\|f\|_2\|g\|_2$$

for every $N \in \mathbb{N}$. Also observe that by the condition over j, we know that

$$(2^{-2j}\lambda)^{-N+2} = 2^{2j(N-2)}\lambda^{-N+2} \leq \lambda^{(N-2)/2-(N-2)} = \lambda^{-(N-2)/2}.$$

Notice that we can get rid of the harmless exponent constants and work only with N. This shows that we can take N as big as we wish to make the exponent of λ very low so that it will have no effect on the bound we obtain in the first summand in (3.66). Let us thus focus on that part. If we were able to prove that

$$\|e^{it\Delta} f_k e^{it\Delta} g_k\|_{L^1_x L^\infty_t(\widetilde{B}_k)} \leq C 2^{-j/2}\lambda^\epsilon(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta})\|f_k\|_2\|g_k\|_2, \tag{3.67}$$

for each $\widetilde{B}_k$, then we could sum in k to bound the first summand in (3.66) by

$$\begin{aligned}
&C2^{-j/2}\lambda^\epsilon(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta})\sum_k \|f_k\|_2\|g_k\|_2 \\
&\qquad \leq C2^{-j/2}\lambda^\epsilon(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta})\left(\sum_k \|f_k\|_2^2\right)^{1/2}\left(\sum_k \|g_k\|_2^2\right)^{1/2} \\
&\qquad \leq C2^{-j/2}\lambda^\epsilon(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta})(2^{-2j}\lambda)^{2\epsilon}\|f\|_2\|g\|_2
\end{aligned}$$

where first we have used Cauchy-Schwarz' inequality and (3.64) afterwards. We can remove the $2^{-2j\epsilon}$ term and merge the λ^ϵ terms. In the end, by joining both parts in (3.66), we get the bound given by

$$C\left(2^{-j/2}\lambda^\epsilon(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta}) + \lambda^{-N}\right)\|f\|_2\|g\|_2.$$

Since $\lambda^{-1/4} < 2^{-j}$, we can write $\lambda^{-N} < 2^{-j/2}\lambda^{-N+1/8}$ and then the bound has terms

$$2^{-j/2}\left(\lambda^{\epsilon}(\lambda^{\alpha(1-\delta)} + \lambda^{3/4+c\delta}) + \lambda^{-N}\right)$$

where λ^{-N} can be bounded by the other term which governs the expression. Hence we see that it is enough to prove (3.67), and for that we have to work on each of the cubes $\widetilde{B}_k$.

From now on we fix k and we focus on $\widetilde{B}_k$, f_k and g_k. To prove (3.67) we will need the wave-packet decomposition analysed in Section 3.1 and a more technical lemma due to T. Tao in [12]. We present the lemma here in the version of [7].

Lemma 3.18. *Let $\lambda \gg 1$ and $0 < \delta \ll 1$. Consider $Q(\lambda) \times [0,\lambda]$ and a partition by means of cubes $\{b_l\}_l$ of length $\lambda^{1-\delta}$. For two functions $f, g \in L^2$ with Fourier support in $Q(1)$ we consider the wave-packet decomposition at scale λ given by $f = \sum_T f_T$ and $g = \sum_T g_T$. If $d(\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g}) \approx 1$, there exists a relation $\sim$ between tubes $T \in \mathcal{T}(\lambda)$ and cubes $\{b_l\}_l$ such that for any $\epsilon > 0$,*

$$\sum_l \left\| \sum_{T\sim b_l} f_T \right\|_2^2 \le C\lambda^{\epsilon}\|f\|_2^2; \qquad \sum_l \left\| \sum_{T\sim b_l} g_T \right\|_2^2 \le C\lambda^{\epsilon}\|g\|_2^2. \tag{3.68}$$

Moreover, if we fix b_l, then for any smooth m_1, m_2 functions supported in $Q(2)$ which generate multiplier operators and $\epsilon > 0$ we have

$$\left\| \sum_{T\nsim b_l \text{ or } T'\nsim b_l} e^{it\Delta}(m_1(D)f_T)e^{it\Delta}(m_2(D)g_{T'}) \right\|_{L^2(b_l)} \le C\lambda^{\epsilon}\lambda^{c\delta-1/4}\|f\|_2\|g\|_2. \tag{3.69}$$

In particular, for trivial multipliers we have

$$\left\| \sum_{T\nsim b_l \text{ or } T'\nsim b_l} e^{it\Delta}f_T e^{it\Delta}g_{T'} \right\|_{L^2(b_l)} \le C\lambda^{\epsilon}\lambda^{c\delta-1/4}\|f\|_2\|g\|_2.$$

Moreover, the sum in $\{T \nsim b_l \text{ or } T' \nsim b_l\}$ can be replaced by the sum in $\{T \nsim b_L, T' \in \mathcal{P}\}$ or by $\{T \in \mathcal{P}, T' \nsim b_L\}$ for any $\mathcal{P} \subset \mathcal{T}(\lambda)$.

We want to apply this lemma to $\widetilde{B}_k, f_k$ and g_k. Observe that $\widetilde{B}_k$ is a space-time cube with side-length $2^{-2j}\lambda$, and that $\operatorname{supp}\widehat{f_k} \subset \operatorname{supp}\widehat{f} + O((2^{-2j}\lambda)^{-1/2}), \operatorname{supp}\widehat{g_k} \subset \operatorname{supp}\widehat{g} + O((2^{-2j}\lambda)^{-1/2})$. We see that $2^j < \lambda^{1/4}$ implies $\lambda^{3/4} \le 2^{-2j}\lambda$, so $(2^{-2j}\lambda)^{-1/2}$ is very small. Also recall that f and g were Fourier supported in $B(1)$, so the first support condition holds for f_k, g_k. On the other hand, since $d(\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g}) \approx 1$, we also have $d(\operatorname{supp}\widehat{f_k}, \operatorname{supp}\widehat{g_k}) \approx 1$ and the lemma can be applied. Thus consider the wave-packet decompositions of f_k and g_k at scale $2^{-2j}\lambda$,

$$f_k = \sum_{T\in\mathcal{T}(2^{-2j}\lambda)} f_{k,T}; \qquad g_k = \sum_{T'\in\mathcal{T}(2^{-2j}\lambda)} g_{k,T'}.$$

We also partition our cube $\widetilde{B}_k$ into cubes of length $(2^{-2j}\lambda)^{1-\delta}$ which we will call $\{\widetilde{b}_{k,l}\}_l$. we can even fix the size slightly so that the union of these cubes is exactly $\widetilde{B}_k$. It is easy to see that there will be $O((2^{-2j}\lambda)^{3\delta})$ small cubes, so that

$$\widetilde{B}_k = \bigcup_l \widetilde{b}_{k,l}. \tag{3.70}$$

Thus, estimates (3.68) and (3.69) are available after substitution of λ by $2^{-2j}\lambda$.

Now by using (3.70) in what we need to prove (3.67), the triangle inequality shows that

$$\|e^{it\Delta}f_k e^{it\Delta}g_k\|_{L^1_x L^\infty_t(\widetilde{B}_k)} \leq \sum_l \|e^{it\Delta}f_k e^{it\Delta}g_k\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})}.$$

We also write the wave-packet decomposition for both f_k and g_k to write

$$\sum_l \|e^{it\Delta}f_k e^{it\Delta}g_k\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})} \leq \sum_l \|\sum_{T,T'\in\mathcal{T}(2^{-2j}\lambda)} e^{it\Delta}f_{k,T}e^{it\Delta}g_{k,T'}\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})}.$$

Notice that having sums in both the cubes $\{\widetilde{b}_{k,l}\}_l$ and the tubes T, we can use the relation in Lemma 3.18 to split the above expression depending if the tubes are reated to $\widetilde{b}_{k,l}$ or not:

$$\begin{aligned}\sum_l \|\sum_{T,T'\in\mathcal{T}(2^{-2j}\lambda)} e^{it\Delta}f_{k,T}e^{it\Delta}g_{k,T'}\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})} &\leq \sum_l \Big\|\sum_{T,T'\sim\widetilde{b}_{k,l}} e^{it\Delta}f_{k,T}e^{it\Delta}g_{k,T'}\Big\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})} \\ &\quad + \sum_l \Big\|\sum_{T\nsim\widetilde{b}_{k,l} \text{ or } T'\nsim\widetilde{b}_{k,l}} e^{it\Delta}f_{k,T}e^{it\Delta}g_{k,T'}\Big\|_{L^1_x L^\infty_t(\widetilde{b}_{k,l})} \\ &= I + II.\end{aligned} \tag{3.71}$$

In other words, I concentrates the tubes related to each $\widetilde{b}_{k,l}$ and II the cases in which some tube is not related to it. We will treat them separately. More precisely, we will prove that for $\epsilon,\delta>0$ there is $c>0$ independent of ϵ and δ such that

$$I \leq C\lambda^\epsilon 2^{-j/2}(\lambda^{1-\delta}2^{2j\delta})^{\alpha+\epsilon}\|f_k\|_2\|g_k\|_2 \tag{3.72}$$

and

$$II \leq C\lambda^\epsilon 2^{-j/2}\lambda^{3/4+c\delta}\|f_k\|_2\|g_k\|_2. \tag{3.73}$$

Notice that since $2^j<\lambda^{1/4}$, we have $2^{2j\delta}<\lambda^{\delta/2}$. Hence, $\lambda^{1-\delta}2^{2j\delta}<\lambda^{1-\delta/2}$. Moreover, since we already have a λ^ϵ term, we bound $\lambda^{\epsilon(1-\delta/2)}<\lambda^\epsilon$. Then (3.72) is bounded by

$$C\lambda^{2\epsilon}2^{-j/2}\lambda^{\alpha(1-\delta/2)}\|f_k\|_2\|g_k\|_2.$$

The sum of (3.72) and (3.73) gives the desired result we sought in (3.67) after renaming $2\epsilon\leftrightarrow\epsilon$ and $\delta/2\leftrightarrow\delta$, so we are left to prove the estimates for I and II.

The proof of I is shorter and will be tackled first. Observe that we look for a bound depending on α, so we will need to use the hypothesis (3.61). Let us define, in the same way we defined B_k out of $\widetilde{B}_k$, the parallelepipeds $b_{k,l}$ by

$$b_{k,l} = \left\{(x,t) \mid (x-2^{j+1}2\pi te_1,t)\in\widetilde{b}_{k,l}\right\}.$$

It is clear that since $\widetilde{B}_k=\bigcup_l \widetilde{b}_{k,l}$ we have

$$B_k = \bigcup_l b_{k,l}.$$

We also decide that the cube-tube relation will be kept. In other words, $b_{k,l} \sim T \Leftrightarrow \widetilde{b}_{k,l} \sim T$.

From the expression of I in (3.71) we fix l and we consider the norm. By the change of variables used to define $b_{k,l}$ we see that I can be written as

$$I = \sum_l \| \sum_{T,T' \sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T}(x - 2^{j+1}2\pi t e_1) e^{it\Delta} g_{k,T'}(x - 2^{j+1}2\pi t e_1) \|_{L^1_x L^\infty_t(b_{k,l})},$$

which is in the shape of the hypothesis (3.61). Nevertheless, the norm needs to be taken in a rectangle, not in a parallelepiped. So we want to consider a rectangle R containing $b_{k,l}$. We know that the height of $b_{k,l}$ is the same as that of $\widetilde{b}_{k,l}$, which is $(2^{-2j}\lambda)^{1-\delta} = 2^{-2j}2^{2j\delta}\lambda^{1-\delta}$. On the other hand, as we said when we defined B_k, the inclined cubes have slope $\approx 2^{-j}$. It is easy to see that the inclination produces an extra length in direction e_1 of $2^{j+1}2\pi(2^{-2j}\lambda)^{1-\delta}$. Then, the total length of the cube in the direction e_1 is

$$(2^{-2j}\lambda)^{1-\delta} + 2^{j+1}2\pi(2^{-2j}\lambda)^{1-\delta} \le 2(2^{-2j}\lambda)^{1-\delta}2\pi 2^{j+1} = 8\pi 2^{-j}2^{2j\delta}\lambda^{1-\delta}.$$

Hence consider $R = Q(x_0, 8\pi 2^{-j}2^{2j\delta}\lambda^{1-\delta}) \times [t_0, t_0 + 4\cdot 2^{-2j}2^{2j\delta}\lambda^{1-\delta}]$ where x_0, t_0 are such that make $b_{k,l} \subset R$. Observe that this way,

$$I \le \sum_l \| \sum_{T,T' \sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T}(x - 2^{j+1}2\pi t e_1) e^{it\Delta} g_{k,T'}(x - 2^{j+1}2\pi t e_1) \|_{L^1_x L^\infty_t(R)},$$

and R has the appropriate sizes to apply (3.61). Also observe that when proving (3.57), there is no problem translating the time interval or the space cube. Indeed, the time played no role and the important fact about the cube was its size. Then, we can say that (3.61) is translation invariant, so substituting λ for $4\cdot 2^{2j\delta}\lambda^{1-\delta}$ we can write

$$I \le C2^{-j/2}(4\cdot 2^{2j\delta}\lambda^{1-\delta})^\alpha \sum_l \Big\| \sum_{T\sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} \Big\|_2 \Big\| \sum_{T'\sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} \Big\|_2.$$

The sum in l can be managed by means of the Cauchy-Schwarz inequality, and together with the estimates in Lemma 3.18 we write

$$\begin{aligned} I &\le C2^{-j/2}(4\cdot 2^{2j\delta}\lambda^{1-\delta})^\alpha \left(\sum_l \Big\| \sum_{T\sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} \Big\|_2^2 \right)^{1/2} \left(\sum_l \Big\| \sum_{T'\sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} \Big\|_2^2 \right)^{1/2} \\ &\le C2^{-j/2}(4\cdot 2^{2j\delta}\lambda^{1-\delta})^\alpha (2^{-2j}\lambda)^\epsilon \|f_k\|_2 \|g_k\|_2 \\ &\le C\lambda^\epsilon(\lambda^{1-\delta}2^{2j\delta})^\alpha 2^{-j/2}\|f_k\|_2\|g_k\|_2. \end{aligned}$$

where in the last inequality we have used the fact that $2^{-2j\epsilon} < 1$. This is the estimate we were looking for in (3.72).

Therefore we are left to prove the estimate for II given in (3.73). We are going to prove that estimate for each of the norms we are summing. In that case we should sum the bound in (3.73) in l. We know that there are $\approx (2^{-2j}\lambda)^{3\delta}$ cubes $b_{k,l}$ for each k, so since the bound does not depend on l, we have

$$II \le C\lambda^\epsilon 2^{-j/2}\lambda^{3/4+c\delta}\|f_k\|_2\|g_k\|_2(2^{-2j}\lambda)^{3\delta} \le C\lambda^\epsilon 2^{-j/2}\lambda^{3/4+(c+3)\delta}\|f_k\|_2\|g_k\|_2.$$

We have used $2^{-2j} < 1$. Hence we would obtain the result with the constant $c^* = c + 3$, which will be independent of ϵ and δ if c is so. Thus we focus on proving

$$II_l = \Big\| \sum_{T \nsim \widetilde{b}_{k,l} \text{ or } T' \nsim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} e^{it\Delta} g_{k,T'} \Big\|_{L_x^1 L_t^\infty(\widetilde{b}_{k,l})} \le C \lambda^\epsilon 2^{-j/2} \lambda^{3/4 + c\delta} \|f_k\|_2 \|g_k\|_2 \tag{3.74}$$

for fixed l. We first decompose the sum into the three possible combinations (that is to say, when T is related and not T', when T' is related and not T and when neither T nor T' is related to $\widetilde{b}_{k,l}$). In other words, we define

$$II_1 = \Big\| \sum_{T \nsim \widetilde{b}_{k,l}, T' \nsim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} e^{it\Delta} g_{k,T'} \Big\|_{L_x^1 L_t^\infty(\widetilde{b}_{k,l})},$$

$$II_2 = \Big\| \sum_{T \nsim \widetilde{b}_{k,l}, T' \sim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} e^{it\Delta} g_{k,T'} \Big\|_{L_x^1 L_t^\infty(\widetilde{b}_{k,l})}$$

and

$$II_3 = \Big\| \sum_{T \sim \widetilde{b}_{k,l}, T' \nsim \widetilde{b}_{k,l}} e^{it\Delta} f_{k,T} e^{it\Delta} g_{k,T'} \Big\|_{L_x^1 L_t^\infty(\widetilde{b}_{k,l})}$$

so that $II_l \le II_1 + II_2 + II_3$. We will see that the three have the same bound, so the bound for II_l will follow with a constant factor equal to 3. The key point is that Lemma 3.18 allows to use the same bound for the three cases, so we will only work with one, II_2.

For simplicity, we define

$$F = \sum_{T \nsim \widetilde{b}_{k,l}} f_{k,T}; \qquad G = \sum_{T' \sim \widetilde{b}_{k,l}} g_{k,T}$$

so that after the usual change of variables we get

$$II_2 = \Big\| e^{it\Delta} F e^{it\Delta} G \Big\|_{L_x^1 L_t^\infty(\widetilde{b}_{k,l})} = \Big\| e^{it\Delta} F(x - 2^{j+1} 2\pi t e_1) e^{it\Delta} G(x - 2^{j+1} 2\pi t e_1) \Big\|_{L_x^1 L_t^\infty(b_{k,l})}.$$

Our next objective is to treat the norms in space and time separately. But observe that $b_{k,l}$ is not a cube any more so it cannot be trivially split. Assume that the original small cube $\widetilde{b}_{k,l}$ has the form

$$\widetilde{b}_{k,l} = Q(y, (2^{-2j}\lambda)^{1-\delta}) \times [s, s + (2^{-2j}\lambda)^{1-\delta}]$$

for some $y = (y_1, y_2) \in \mathbb{R}^2$, $s \in \mathbb{R}$. We have only fixed what we could call the centre of the cube. Once we have done this, let us call Q_0 to the projection of $b_{k,l}$ in space. This is to say,

$$Q_0 = \{x \in \mathbb{R}^2 \mid (x,t) \in b_{k,l} \text{ for some } t\}.$$

For each $x \in Q_0$, it is rather clear that the time values t such that $(x,t) \in b_{k,l}$ form an interval. Let us call that interval

$$I_x = \{t \in \mathbb{R} \mid (x,t) \in b_{k,l}\} = (a_-(x), a_+(x)).$$

This allows us to write

$$II_2 = \Big\| \sup_{t \in I_x} \Big| e^{it\Delta} F(x - 2^{j+1} 2\pi t e_1) e^{it\Delta} G(x - 2^{j+1} 2\pi t e_1) \Big| \Big\|_{L_x^1(Q_0)}.$$

For the management of the supremum we recall Lemma 3.9. It requires to work with the lower bound of the interval, $a_-(x)$, which we will need to specify. It is given by a straight line of slope $2^{-j}/4\pi$ in direction e_1 which goes through the right border of the cube $y_1 + (2^{-2j}\lambda)^{1-\delta}/2$. It is also important to notice that it cannot go below s, so we have

$$a_-(x) = \max\left\{ s, \frac{2^{-j}}{4\pi}(x_1 - y_1 - (2^{-2j}\lambda)^{1-\delta}/2) \right\}. \tag{3.75}$$

Hence, we apply Lemma 3.9 with $\mu = 2^j$ to obtain

$$\begin{aligned} \sup_{t\in I_x} &\left| e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right| \\ &\lesssim \left| e^{ia_-(x)\Delta}F(x - 2^{j+1}2\pi a_-(x)e_1) e^{ia_-(x)\Delta}G(x - 2^{j+1}2\pi a_-(x)e_1) \right| \\ &+ 2^{j/2} \left\| e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right\|_{L^2(I_x)} \\ &+ 2^{-j/2} \left\| \partial_t \left(e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right) \right\|_{L^2(I_x)}. \end{aligned} \tag{3.76}$$

Let us manage the derivatives. Observe that by the Schwartz regularity of the functions we are working with, we can take the derivative inside the integral to say that

$$\partial_t \left(e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) \right) = -4\pi^2 i \int_{\mathbb{R}^2} \widehat{F}(\xi)(|\xi|^2 + 2^{j+1}e_1\cdot\xi) e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2 - 2^{j+1}2\pi t e_1\cdot\xi)}\, d\xi.$$

If we recall the differentiation properties of the Fourier transform, together with the absolute value, we see that

$$\begin{aligned} \left| \partial_t \left(e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) \right) \right| &= \left| \int_{\mathbb{R}^2} (\widehat{\Delta F}(\xi) - 2^{j+1}2\pi \widehat{D_1 F}(\xi)) e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2 - 2^{j+1}2\pi t e_1\cdot\xi)}\, d\xi \right| \\ &= \left| e^{it\Delta}(\Delta F - 2\pi 2^{j+1} D_1 F)(x - 2^{j+1}2\pi t e_1) \right|, \end{aligned}$$

where $D_1 F = \partial_{x_1} F$ is the partial derivative in direction e_1. Hence, observe that the derivation has generated a multiplier operator, which is given by $m(\xi) = 4\pi^2(|\xi|^2 + 2^{j+1}\xi_1)$. The differential operator it generates in the solution is $\overline{m}(D) = \Delta - 2^{j+1}2\pi D_1$. Then, by the usual product rule in (3.76), we can write

$$\sup_{t\in I_x} \left| e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right| \lesssim \Omega_-(F,G)(x) + 2^{j/2} \sum_{i=1}^{3} \Omega_i(F,G)(x) \tag{3.77}$$

where

$$\begin{aligned} \Omega_-(F,G)(x) &= \left| e^{ia_-(x)\Delta}F(x - 2^{j+1}2\pi a_-(x)e_1) e^{ia_-(x)\Delta}G(x - 2^{j+1}2\pi a_-(x)e_1) \right|, \\ \Omega_1(F,G)(x) &= \left\| e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right\|_{L^2(I_x)}, \\ \Omega_2(F,G)(x) &= \left\| e^{it\Delta}(2^{-j}\Delta F - 4\pi D_1 F)(x - 2^{j+1}2\pi t e_1) e^{it\Delta}G(x - 2^{j+1}2\pi t e_1) \right\|_{L^2(I_x)}, \\ \Omega_3(F,G)(x) &= \left\| e^{it\Delta}F(x - 2^{j+1}2\pi t e_1) e^{it\Delta}(2^{-j}\Delta G - 4\pi D_1 G)(x - 2^{j+1}2\pi t e_1) \right\|_{L^2(I_x)}, \end{aligned}$$

where to write the last two terms we have multiplied a 2^{-j} term so that in (3.77) we can have a $2^{j/2}$ factor outside the sum. By virtue of this, we have

$$II_2 \lesssim \|\Omega_-(F,G)\|_{L^1(Q_0)} + 2^{j/2}\sum_{i=1}^{3}\|\Omega_i(F,G)\|_{L^1(Q_0)}, \tag{3.78}$$

so our objective is to bound every L^1-norm above. We start by the estimates of Ω_i, $i=1,2,3$. Observe that the multipliers will not affect when applying the estimates (3.69) seen in Lemma 3.18, so we can manage the three cases in the same way. Consider for instance the case of Ω_2. If we write the original meaning of F and G we see that

$$\begin{aligned}&\|\Omega_2(F,G)\|_{L^1(Q_0)}\\ &\quad= \Big\|\sum_{T\nsim b_{k,l},T'\sim b_{k,l}} e^{it\Delta}(2^{-j}\Delta-4\pi D_1)f_{k,T}(x-2^{j+1}2\pi te_1)e^{it\Delta}g_{k,T'}(x-2^{j+1}2\pi te_1)\Big\|_{L^1_xL^2_t(b_{k,l})},\end{aligned}$$

because $b_{k,l}=\bigcup_{x\in Q_0}\{x\}\times I_x$. Now we use Hölder's inequality in space, in Q_0. This way we can write the L^2_x-norm by also considering $|Q_0|^{1/2}$. More precisely,

$$|Q_0| = (2^{-2j}\lambda)^{1-\delta}(1+2\pi 2^{j+1})(2^{-2j}\lambda)^{1-\delta} \approx 2\pi 2^{j+1}(2^{-2j}\lambda)^{2(1-\delta)},$$

so if we call $\widetilde{f_{k,T}} = (2^{-j}\Delta-4\pi D_1)f_{k,T}$, we have

$$\begin{aligned}&\|\Omega_2(F,G)\|_{L^1(Q_0)}\\ &\quad\le C2^{j/2}(2^{-2j}\lambda)^{1-\delta}\Big\|\sum_{T\nsim b_{k,l},T'\sim b_{k,l}} e^{it\Delta}\widetilde{f_{k,T}}(x-2^{j+1}2\pi te_1)e^{it\Delta}g_{k,T'}(x-2^{j+1}2\pi te_1)\Big\|_{L^2_{x,t}(b_{k,l})}.\end{aligned}$$

Now we can use the estimates in Lemma 3.18 to say that

$$\begin{aligned}\|\Omega_2(F,G)\|_{L^1(Q_0)} &\le C2^{j/2}(2^{-2j}\lambda)^{1-\delta}(2^{-2j}\lambda)^{\epsilon+c\delta-1/4}\|f_k\|_2\|_k\|_2\\ &= C2^{j/2}(2^{-2j}\lambda)^{\epsilon+(c-1)\delta+3/4}\|f_k\|_2\|_k\|_2.\end{aligned} \tag{3.79}$$

We can see that since $1\le 2^j<\lambda^{1/4}$, then $1<\lambda 2^{-4j}<\lambda 2^{-2j}$ and hence (3.79) is bounded by

$$C2^{j/2}(2^{-2j}\lambda)^{\epsilon+c\delta+3/4}\|f_k\|_2\|_k\|_2 \le C2^{-j}\lambda^{\epsilon+c\delta+3/4}\|f_k\|_2\|_k\|_2,$$

where we have removed $(2^{-2j})^{\epsilon+c\delta}\le 1$. Therefore, observing that the procedure is the same for Ω_1 and Ω_3 (in one case there are no multipliers and in the other case we define $\widetilde{g_{k,T}}$ the same way we have defined $\widetilde{f_{k,T}}$), we have the estimates

$$\|\Omega_i(F,G)\|_{L^1(Q_0)} \le C2^{-j}\lambda^{\epsilon+c\delta+3/4}\|f_k\|_2\|_k\|_2, \qquad \forall i=1,2,3. \tag{3.80}$$

Thus we are left to prove the estimate for $\Omega_-(F,G)$. For this we present a lemma.

Lemma 3.19. *Let $\lambda\ge 1$ and $a_-(x)$ be the bound of I_x defined in* (3.75). *Consider a function f with Fourier support in $Q(1)$ and $x_0\in\mathbb{R}^2$. Then, there exists a constant $C>0$ independent of x_0, s and y (recall that $a_-(x)$ depends on s,y) such that*

$$\|e^{ia_-(x)\Delta}f(x-2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q(x_0,\lambda))} \le C\lambda^{1/2}\|f\|_2.$$

Moreover, if in the support of $\widehat{f}$ we have $|\xi_1| \approx 1$, then

$$\|e^{ia_-(x)\Delta} f(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q(x_0,\lambda))} \leq C2^{j/2}\|f\|_2.$$

Let us prove this lemma. We easily see that the phase of the integral expression of $e^{ia_-(x)\Delta}f(x - 2^{j+1}2\pi a_-(x)e_1)$ is $2\pi(x\cdot\xi - 2\pi a_-(x)|\xi|^2 - 2^{j+1}2\pi a_-(x)\xi_1)$, which depending on the value of $a_-(x)$ can be

1. $2\pi(x\cdot\xi - 2\pi s(|\xi|^2 + 2^{j+1}\xi_1))$, or
2. $2\pi\left(x\cdot\xi - 2\pi\frac{2^{-j}}{4\pi}(x_1 - y_1 - (2^{-2j}\lambda)^{1-\delta}/2)(|\xi|^2 + 2^{j+1}\xi_1)\right)$.

Observe that in case 1 we have

$$e^{ia_-(x)\Delta}f(x - 2^{j+1}2\pi a_-(x)e_1) = \mathscr{F}^{-1}\left(\widehat{f}(\xi)e^{-4\pi^2 is(|\xi|^2+2^{j+1}\xi_1)}\right) = f_1,$$

where by Plancherel's identity $\|f_1\|_2 = \|f\|_2$. On the other hand, if we are in case 2, we can write

$$\begin{aligned} e^{ia_-(x)\Delta}f(x - 2^{j+1}2\pi a_-(x)e_1) &= \int_{\mathbb{R}^2}\widehat{f}(\xi)e^{2\pi i(x\cdot\xi - 2\pi\frac{2^{-j}}{4\pi}(x_1 - y_1 - (2^{-2j}\lambda)^{1-\delta}/2)(|\xi|^2+2^{j+1}\xi_1))}\,d\xi \\ &= \int_{\mathbb{R}^2}\widehat{f_2}(\xi)e^{2\pi i(x\cdot\xi - 2\pi\frac{2^{-j}}{4\pi}x_1(|\xi|^2+2^{j+1}\xi_1))}\,d\xi \\ &:= Tf_2(x), \end{aligned} \tag{3.81}$$

where $\widehat{f_2}(\xi) = \widehat{f}(\xi)e^{4\pi^2 i\frac{2^{-j}}{4\pi}(y_1+(2^{-2j}\lambda)^{1-\delta}/2)(|\xi|^2-2^{j+1}\xi_1))}$. Obviously $\|f_2\|_2 = \|f\|_2$.

We can manage the two cases at the same time if we sum both norms, so that

$$\|e^{ia_-(x)\Delta}f(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q(x_0,\lambda))} \leq \|f_1\|_2 + \|Tf_2\|_2 = \|f\|_2 + \|Tf_2\|_2. \tag{3.82}$$

So we need to work on the operator T. Observe that the phase in its definition in (3.81) can be simplified so that

$$x\cdot\xi - 2\pi\frac{2^{-j}}{4\pi}x_1(|\xi|^2 + 2^{j+1}\xi_1) = x_1\xi_1 + x_2\xi_2 - 2^{-j-1}x_1|\xi|^2 - x_1\xi_1 = x_2\xi_2 - 2^{-j-1}x_1|\xi|^2.$$

Observe that we can work with the Fourier transform in the second variable, since

$$Tf_2(x) = \int_{\mathbb{R}^2}\widehat{f_2}(\xi)e^{-2\pi i 2^{-j-1}x_1|\xi|^2}e^{2\pi i x_2\xi_2}\,d\xi = \mathscr{F}^{-1}_{\xi_2}\left(\int_{\mathbb{R}}\widehat{f_2}(\xi)e^{-2\pi i 2^{-j-1}x_1|\xi|^2}\,d\xi_1\right)(x_2).$$

Plancherel's identity says that the norm in the second variable does not change, so

$$\|Tf_2\|^2_{L^2_{x_2}} = \left\|\int_{\mathbb{R}}\widehat{f_2}(\xi)e^{-2\pi i 2^{-j-1}x_1|\xi|^2}\,d\xi_1\right\|^2_{L^2_{\xi_2}}.$$

Observe that by the triangle integral inequality we can write

$$\left\|\int_{\mathbb{R}}\widehat{f_2}(\xi)e^{-2\pi i 2^{-j-1}x_1|\xi|^2}\,d\xi_1\right\|^2_{L^2_{\xi_2}} \leq \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\widehat{f_2}(\xi)|\,d\xi_1\right)^2 d\xi_2.$$

Recall that $\operatorname{supp}\widehat{f_2} = \operatorname{supp}\widehat{f} \subset Q(1)$, so in the inner integral we are only integrating in $(-1,1)$. Hence by Hölder's inequality we can say that

$$\left(\int_{\mathbb{R}} |\widehat{f_2}(\xi)|\, d\xi_1\right)^2 \leq C\|\widehat{f_2}\|^2_{L^2_{\xi_2}},$$

so we get $\|Tf_2\|^2_{L^2_{x_2}} \leq C\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{f_2}(\xi)|^2\, d\xi_1\, d\xi_2 = \|\widehat{f_2}\|_2^2$. Now recall that we were interested in the norms in $Q(x_0,\lambda)$, so

$$\|Tf_2\|^2_{L^2(Q(x_0,\lambda)} \leq C\int_{x_0-\lambda/2}^{x_0+\lambda/2} \|Tf_2\|^2_{L^2_{x_2}}\, dx_1 \leq C\lambda\|\widehat{f_2}\|_2^2.$$

If we recall that $\|\widehat{f_2}\|_2 = \|\widehat{f}\|_2$, we see that we have

$$\|Tf_2\|_{L^2(Q(x_0,\lambda)} \leq C\lambda^{1/2}\|\widehat{f}\|_2,$$

and in all, going back to (3.82) we see that

$$\|e^{ia_-(x)\Delta}f(x-2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q(x_0,\lambda))} \leq \|f\|_2 + C\lambda^{1/2}\|\widehat{f}\|_2 \leq C\lambda^{1/2}\|f\|_2$$

as desired.

Now suppose that we have $|\xi_1| \approx 1$ in $\operatorname{supp}\widehat{f}$. In that case, we do the change of variables $(\eta_1,\eta_2) = (-2^{-j-1}|\xi|^2, \xi_2)$. Then we have

$$Tf_2(x) = \int_{\mathbb{R}^2} \widehat{f_2}(\xi_1(\eta_1,\eta_2),\eta_2)e^{2\pi i(x_1\eta_1+x_2\eta_2)}\cdot|J|\, d\eta.$$

We know that the Jacobian can be represented by $d\xi = \left|\partial\xi/\partial\eta\right| d\eta = |J| d\eta$. But in this case it is easier for us to work out $\left|\partial\eta/\partial\xi\right|$. It is easy to check that

$$\left|\frac{\partial\eta}{\partial\xi}\right| = 2^{-j}\xi_1.$$

Since the Jacobian of the change we have performed is the inverse of the above expression, we get

$$Tf_2(x) = 2^j\int_{\mathbb{R}^2} \widehat{f_2}(\xi_1(\eta_1,\eta_2),\eta_2)e^{2\pi i(x_1\eta_1+x_2\eta_2)}\frac{1}{|\xi_1(\eta_1,\eta_2)|}\, d\eta.$$

Call $\widehat{g}(\eta_1,\eta_2) = \widehat{f_2}(\xi_1(\eta_1,\eta_2),\eta_2)/|\xi_1(\eta_1,\eta_2)|$ so that

$$Tf_2(x) = 2^j\mathscr{F}^{-1}(\widehat{g}(\eta_1,\eta_2))(x).$$

By Plancherel we have

$$\|Tf_2\|_2 = 2^j\|\widehat{g}\|_2.$$

Next step is to revert the change of variables. Indeed,

$$\|\widehat{g}\|_2^2 = \int_{\mathbb{R}^2}\left|\frac{\widehat{f_2}(\xi_1,\eta_2)}{|\xi_1|}\right|^2 d\eta = \int_{\mathbb{R}^2}\left|\frac{\widehat{f_2}(\xi_1,\xi_2)}{|\xi_1|}\right|^2 2^{-j}|\xi_1|d\xi = 2^{-j}\int_{\mathbb{R}^2}\frac{|\widehat{f_2}(\xi)|^2}{|\xi_1|}d\xi.$$

By hypothesis, $|\xi_1| \approx 1$, so we deduce $\|Tf_2\|_2 = 2^j 2^{-j/2}\|f_2\|_2$. Since we know that f_2 has the same L^2-norm as f, we conclude that

$$\|Tf_2\|_2 = 2^{j/2}\|f\|_2,$$

which is enough to assert that

$$\|e^{ia_-(x)\Delta} f(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q(x_0,\lambda))} \leq \|f\|_2 + 2^{j/2}\|f\|_2 \leq C2^{j/2}\|f\|_2$$

and the lemma is proven.

Recall we wanted the lemma to obtain the estimate of Ω_-. Now we can do it. Indeed, since we need to estimate the $L^1(Q_0)$-norm, we use Hölder's inequality to obtain L^2-norms, so

$$\|\Omega_-(F,G)\|_{L^1(Q_0)} \leq \|e^{ia_-(x)\Delta}F(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q_0)}\, \|e^{ia_-(x)\Delta}G(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q_0)}.$$

We will apply a consequence of Lemma 3.19 to each term. Observe that we are working with supports in $B(1)$ and $d(\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g}) \approx 1$. That means that both supports cannot be arbitrarily close. A consequence of this is that it cannot happen that both of them are close to zero. We can also make one of them not to be lying on the vertical axis, so that $|\xi_1| \approx 1$. Suppose it is the case of $\widehat{f}$. Then, since F is defined by a sum of wave-packets of f_k, $\operatorname{supp}\widehat{F}$ is a little perturbation of $\operatorname{supp}\widehat{f}$, so we can assume $|\xi_1| \approx 1$ there. Hence, by Lemma 3.19,

$$\|e^{ia_-(x)\Delta}F(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q_0)} \leq C2^{j/2}\|F\|_2.$$

The lemma can be applied since we are integrating on a rectangle Q_0 whose largest side is $(2^{-2j}\lambda)^{1-\delta}2^{j+1} > 1$. On the other hand, we apply the first consequence to G to obtain

$$\|e^{ia_-(x)\Delta}G(x - 2^{j+1}2\pi a_-(x)e_1)\|_{L^2(Q_0)} \leq C((2^{-2j}\lambda)^{1-\delta}2^{j+1})^{1/2}\|G\|_2.$$

Joining both estimates together we see that

$$\|\Omega_-(F,G)\|_{L^1(Q_0)} \leq C2^{j/2}2^{-j}2^{j\delta}\lambda^{(1-\delta)/2}2^{(j+1)/2}\|F\|_2\|G\|_2.$$

On the one hand, save the norms, we obtain a bound of

$$C2^{j/2}2^{(j+1)/2}2^{-j}2^{j\delta}\lambda^{1/2}\lambda^{-\delta/2} = C2^{j\delta}\lambda^{1/2}\lambda^{-\delta/2} \leq C2^{j\delta}\lambda^{1/2}.$$

On the other hand, the norms can be controlled by the results of Lemma 3.5, and more precisely by means of the Plancherel-type equivalences we proved in (3.33). Indeed,

$$\|F\|_2^2 = \Big\|\sum_{T\sim b_{k,l}} f_{k,T}\Big\|_2^2 \approx \sum_{T\sim b_{k,l}} \|f_{k,T}\|_2^2 \leq \sum_T \|f_{k,T}\|_2^2 \approx \|f_k\|_2^2,$$

and the same follows for G and g_k. Hence, we have shown that

$$\|\Omega_-(F,G)\|_{L^1(Q_0)} \leq C2^{j\delta}\lambda^{1/2}\|f_k\|_2\|g_k\|_2. \tag{3.83}$$

With the estimates for Ω_i (3.80) and (3.83) at hand, we go back to (3.78) and we see that

$$II_2 \leq C\left(2^{j\delta}\lambda^{1/2} + 2^{j/2}2^{-j}\lambda^{\epsilon+c\delta+3/4}\right)\|f_k\|_2\|g_k\|_2 = C\left(2^{j\delta}\lambda^{1/2} + 2^{-j/2}\lambda^{\epsilon+c\delta+3/4}\right)\|f_k\|_2\|g_k\|_2.$$

Observe that $2^{j\delta}\lambda^{1/2} < \lambda^{\delta/4+1/2} = \lambda^{3/4+\delta/4}\lambda^{-1/4} < 2^{-j}\lambda^{3/4+\delta/4}$, so

$$II_2 \leq C\left(2^{-j}\lambda^{3/4+\delta/4} + 2^{-j/2}\lambda^{\epsilon+c\delta+3/4}\right)\|f_k\|_2\|g_k\|_2 \leq C2^{-j/2}\lambda^{\epsilon}\lambda^{3/4+c'\delta}\|f_k\|_2\|g_k\|_2,$$

with $c' = \max\{c, 1/4\}$, which is precisely what we asked for in (3.74). We can go through the argument we have performed for II_2 to see that it is valid for II_1 and II_3 because Lemma 3.18 allows the same bounds. We are therefore done. □

A

Oscillatory Integrals of the First Kind

Definition A.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ be smooth functions and $(a,b) \subset \mathbb{R}$ an interval . The function defined by

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx, \quad \lambda > 0 \tag{A.1}$$

is called a **oscillatory integral of the first kind**. The function ϕ is called the **phase** of the integral.

The main objective is to analyse the behaviour of the integral $I(\lambda)$ when λ tends to infinity. We will here treat two phenomena of these integrals: the localisation and the scaling principles.

A.1 Localisation

To deal with the asymptotic behaviour of the integral, we assume first that ψ has compact support in (a,b). In this case, the asymptotic behaviour of $I(\lambda)$ is determined by the points in which $\phi'(x) = 0$. As we will see, the main argument is integration by parts.

Proposition A.2. *Let ϕ, ψ be smooth functions so that ψ has compact support in (a,b) and $\phi'(x) \neq 0$ in (a,b). Then,*

$$I(\lambda) = O(\lambda^{-N}) \quad \textit{as } \lambda \to \infty$$

for every $N \in \mathbb{N}$.

Proof. Define the differential operator

$$Df(x) = \left(i\lambda\phi'(x)\right)^{-1}\frac{df}{dx},$$

which is well defined because $\phi'(x) \neq 0$. If we work with the inner product of $L^2((a,b))$, we know that the transpose (or adjoint) operator, which we will denote by D^t, must satisfy

$$\langle f, Dg\rangle = \langle D^t f, g\rangle. \tag{A.2}$$

Observe that

$$\langle f, Dg\rangle = \frac{1}{i\lambda}\int_a^b f(x)\frac{g'(x)}{\phi'(x)}\,dx,$$

and integration by parts gives

$$\int_a^b f(x)\frac{g'(x)}{\phi'(x)}\,dx = g(x)\frac{f(x)}{\phi'(x)}\Big|_a^b - \int_a^b g(x)\frac{d}{dx}\left(\frac{f(x)}{\phi'(x)}\right)dx.$$

If we consider that either f or g is compactly supported in (a,b) (related to the hypothesis of ψ), then the boundary terms disappear and

$$\langle f, Dg\rangle = -\frac{1}{i\lambda}\int_a^b g(x)\frac{d}{dx}\left(\frac{f(x)}{\phi'(x)}\right)dx.$$

Hence, the transpose operator (as we have seen, only under certain restrictions for the functions) must be

$$D^t f(x) = -\frac{d}{dx}\left(\frac{f(x)}{i\lambda\phi'(x)}\right). \tag{A.3}$$

We observe that if we apply D to the exponential term involving the phase in (A.1),

$$D(e^{i\lambda\phi(x)}) = \frac{e^{i\lambda\phi(x)}i\lambda\phi'(x)}{i\lambda\phi'(x)} = e^{i\lambda\phi(x)}$$

and thus $D^N(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}$ for all $N \in \mathbb{N}$. Therefore, in (A.2) consider $f = e^{i\lambda\phi(x)}$ and $g = \psi$, which has compact support. Then,

$$I(\lambda) = \langle e^{i\lambda\phi}, \psi\rangle = \langle D^N(e^{i\lambda\phi}), \psi\rangle = \langle e^{i\lambda\phi}, (D^t)^N(\psi)\rangle$$

and hence

$$|I(\lambda)| \leq \int_a^b |(D^t)^N(\psi(x))|\,dx.$$

We need to know something more about $(D^t)^N$. But observe that from (A.3) we deduce

$$(D^t)^2\psi = -\frac{1}{i\lambda}\frac{d}{dx}\left(\frac{-\frac{1}{i\lambda}\frac{d}{dx}\left(\frac{\psi}{\phi'}\right)}{\phi'}\right) = \left(-\frac{1}{i\lambda}\right)^2\frac{d}{dx}\left(\frac{1}{\phi'}\frac{d}{dx}\left(\frac{\psi}{\phi'}\right)\right),$$

and since ψ and ϕ are smooth, it is clear that

$$(D^t)^N\psi = \left(-\frac{1}{i\lambda}\right)^N h_N,$$

where h_N is a smooth function depending on the derivatives of ψ and ϕ and hence bounded in (a,b), say by B_N. Therefore, $|(D^t)^N\psi(x)| \leq B_N|\lambda|^{-N}$ and

$$|I(\lambda)| \leq (b-a)B_N|\lambda|^{-N} = A_N|\lambda|^{-N},$$

where $A_N = (b-a)B_N$, which gives the desired result. □

Remark A.3. Observe that if $\phi(x) = x$, then for compactly supported ψ,

$$I(\lambda) = \int_a^b \psi(x) e^{i\lambda x}\, dx = \mathscr{F}\psi(\lambda)$$

is nothing but the Fourier transform of ψ, and Proposition A.2 asserts that it is a function of rapid descent (though not in the sense of the derivatives as in the Schwartz space, even if we know more, that it is indeed a Schwartz function for being the Fourier transform of a compactly supported function).

Remark A.4. If the condition of compactness for the support of ψ is removed, then D and D^t defined above are no longer transposes of each other. In fact, it is necessary to consider the boundary terms, and we get

$$\langle e^{i\lambda\phi}, \psi\rangle = \langle De^{i\lambda\phi}, \psi\rangle = \frac{1}{i\lambda} \frac{e^{i\lambda\phi(x)}\psi(x)}{\phi'(x)}\Big|_a^b + \langle e^{i\lambda\phi}, D^t\psi\rangle.$$

Therefore, iterating we obtain

$$\langle e^{i\lambda\phi}, \psi\rangle = \langle e^{i\lambda\phi}, (D^t)^N\psi\rangle + \frac{1}{i\lambda}\sum_{k=0}^{N-1} \frac{e^{i\lambda\phi}(D^t)^k\psi}{\phi'},$$

which shows that the best we can get is $I(\lambda) = O(\lambda^{-1})$. This fact is clearly shown if $\psi = 1$ and $\phi(x) = x$, since

$$I(\lambda) = \int_a^b e^{i\lambda x}\, dx = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}.$$

Remark A.5. As a last remark, we can say that even if the support is not compact, if both ϕ and ψ and their derivatives up to order N have periodic boundary conditions, this is $\phi^{(k)}(a) = \phi^{(k)}(b)$ and the same for ψ for $k = 0, 1, \ldots, N-1$, then the boundary terms will clearly disappear and the main result will hold, having an estimate of λ^{-N}.

A.2 Scaling

We now suppose that the only information we have is $|\phi^{(k)}(x)| \geq 1$ for some $k \in \mathbb{N}$. We want to obtain an estimate for the oscillatory integral with $\psi \equiv 1$,

$$\int_a^b e^{i\lambda\phi(x)}\, dx,$$

which does not depend on the interval (a, b). The following results are usually referred to as **Van der Corput estimates**.

Proposition A.6. *Let ϕ be a smooth and real valued function in (a, b). Suppose that $|\phi^{(k)}(x)| \geq 1$ for some $k \in \mathbb{N}$ and for all $x \in (a, b)$. Then,*

$$\left|\int_a^b e^{i\lambda\phi(x)}\, dx\right| \leq c_k \lambda^{-1/k}$$

if

1. *$k \geq 2$, or*

2. *$k = 1$ and ϕ' is monotonic.*

Moreover, the constant c_k is independent of ϕ, λ and (a,b).

Remark A.7. We see that the case $k = 1$ is special and needs a different treatment. Indeed, it is not enough to ask $|\phi'(x)| \geq 1$. Consider $\lambda = 1$ and suppose $\phi' \geq 1$ (since ϕ' is continuous, the hypothesis makes it either positive or negative. We consider it is positive and thus ϕ is increasing). Suppose also that ϕ' oscillates, so that it is big when $\cos\phi(x) < 0$ and it is relatively small when $\cos\phi(x) > 0$ (we can also see this as $\phi' \approx 1$ when $\phi(x) \in \ldots \cup (-\pi/2, \pi/2) \cup (3\pi/2, 5\pi/2) \cup \ldots$ and $\phi' \gg 1$ when $\phi(x) \in \ldots \cup (\pi/2, 3\pi/2) \cup (5\pi/2, 7\pi/2) \ldots$). In this case,

$$m(\{x | \cos\phi(x) > 0\}) \gg m(\{x | \cos\phi(x) < 0\}),$$

and thus

$$\mathrm{Re} \int_a^b e^{i\phi(x)}\, dx = \int_a^b \cos\phi(x)\, dx \to \infty \qquad \text{when } b \to \infty.$$

This is the reason for which we need to ask the derivative to be monotonic, so that it does not oscillate. Observe that when $k = 2$, these oscillations cannot occur since $|\phi''(x)| \geq 1$ asserts precisely that ϕ' is monotonic.

Proof. We first prove case 2. Consider the operator D of the proof of Proposition A.2. Then, as we have seen in Remark A.4, and with $\psi \equiv 1$ in this case,

$$\langle e^{i\lambda\phi}, 1\rangle = \langle De^{i\lambda\phi}, 1\rangle = \langle e^{i\lambda\phi}, D^t(1)\rangle + \frac{1}{i\lambda} \frac{e^{i\lambda\phi(x)}}{\phi'(x)} \Big|_a^b. \tag{A.4}$$

The second term in (A.4) can be bounded as follows:

$$\left| \frac{1}{i\lambda} \frac{e^{i\lambda\phi(x)}}{\phi'(x)} \Big|_a^b \right| \leq \frac{1}{\lambda} \left(\frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} \right) \leq \frac{2}{\lambda}$$

because $|\phi'(x)| \geq 1$. On the other hand,

$$|\langle e^{i\lambda\phi}, D^t(1)\rangle| = \left| \int_a^b e^{i\lambda\phi(x)} \frac{d}{dx} \left(\frac{1}{i\lambda\phi'(x)} \right) \right| \leq \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx.$$

Since ϕ' is monotonic, its inverse is so too, and hence $\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)$ does not change sign. This means that we can take the absolute value outside the integral to write

$$|\langle e^{i\lambda\phi}, D^t(1)\rangle| \leq \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \leq \frac{1}{\lambda} \left(\frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} \right) \leq \frac{2}{\lambda}.$$

Therefore, (A.4) can be bounded by

$$\langle e^{i\lambda\phi}, 1\rangle \leq \frac{2}{\lambda} + \frac{2}{\lambda} = \frac{4}{\lambda},$$

and the property holds with $c_1 = 4$.

Case 1 is proven by induction on k. So consider $|\phi^{(k+1)}(x)| \geq 1$. We can assume that $\phi^{(k+1)}(x) \geq 1$ (by continuity it is either positive or negative. If it were negative, the argument is analogous). Hence, $\phi^{(k)}$ is increasing in (a,b), so there are two options:

i There exists $z \in (a,b)$ such that $\phi^{(k)}(z) = 0$, so $|\phi^{(k)}(z)|$ is decreasing in (a,z) and increasing in (z,b). Hence, it has a unique minimum in z.

ii The function $\phi^{(k)}$ is not zero in any point. In that case, it is either positive or negative, and hence the unique minimum of $|\phi^{(k)}|$ is in a or in b.

We treat both cases separately.

i We know $\phi^{(k)}(z) = 0$, and since $\phi^{(k+1)}(x) \geq 1$, then $|\phi^{(k)}|$ increases or decreases faster than x, so for $\delta > 0$,

$$|\phi^{(k)}(x)| \geq \delta, \quad \forall x \notin (z-\delta, z+\delta).$$

Then, by induction, since the property is true for k, we can use it in the intervals $(a, z-\delta)$ and $(z+\delta, b)$, because $|\phi^{(k)}(x)|/\delta \geq 1$ there. Hence,

$$\left|\int_a^{z-\delta} e^{i\lambda\phi(x)}\,dx\right| = \left|\int_a^{z-\delta} e^{i(\lambda\delta)\frac{\phi(x)}{\delta}}\,dx\right| \leq c_k(\lambda\delta)^{-1/k}.$$

In the same way,

$$\left|\int_{z+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| = \left|\int_{z+\delta}^{b} e^{i(\lambda\delta)\frac{\phi(x)}{\delta}}\,dx\right| \leq c_k(\lambda\delta)^{-1/k}.$$

Also,

$$\left|\int_{z-\delta}^{z+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq z+\delta-(z-\delta) = 2\delta.$$

Hence,

$$\left|\int_a^{b} e^{i\lambda\phi(x)}\,dx\right| \leq 2\delta + 2c_k(\lambda\delta)^{-1/k}.$$

ii Consider for simplicity that the minimum is in a. Then, by the same argument, $|\phi^{(k)}(x)| \geq \delta$ when $x \notin (a, a+\delta)$, and by induction,

$$\left|\int_{a+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| = \left|\int_{a+\delta}^{b} e^{i(\lambda\delta)\frac{\phi(x)}{\delta}}\,dx\right| \leq c_k(\lambda\delta)^{-1/k}.$$

On the other hand,

$$\left|\int_a^{a+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq \delta.$$

Therefore,

$$\left|\int_a^{a+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq \delta + c_k(\lambda\delta)^{-1/k}.$$

If the minimum is in b, the same argument works.

We see that in both cases, we obtain a bound of $2\delta + 2c_k(\lambda\delta)^{-1/k}$. Since we want a $\lambda^{-1/(k+1)}$ bound, let us choose an appropriate δ. Indeed,

$$(\lambda\delta)^{-1/k} = \lambda^{-1/(k+1)} \Leftrightarrow \delta^{1/k} = \lambda^{1/(k+1)-1/k} = \lambda^{-1/k(k+1)}.$$

Thus, we need $\delta = \lambda^{-1/(k+1)}$, so that

$$2\delta + 2c_k(\lambda\delta)^{-1/k} = 2\lambda^{-1/(k+1)} + 2c_k\lambda^{-1/(k+1)} = 2(1+c_k)\lambda^{-1/(k+1)}.$$

Therefore, it is enough to choose $c_{k+1} = 2(1+c_k)$. □

Remark A.8. Observe that as we have said before, for the case $k+1=2$, we are able to use case $k=1$ because the fact that $|\phi''| \geq 1$ implies that ϕ' is either increasing or decreasing.

The result in Proposition A.6 allows us to give a similar estimate also when ψ is included.

Corollary A.9. *Let ϕ be a smooth real-valued function on (a,b) which satisfies $|\phi^{(k)}(x)| \geq 1$ in (a,b) for some $k \in \mathbb{N}$. In case $k=1$, we also assume that ϕ' is monotonic. Then, if ψ is a smooth (and possibly complex-valued) function,*

$$|I(\lambda)| \leq c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right],$$

where $I(\lambda)$ is the oscillatory integral defined in (A.1).

Proof. Define

$$F(x) = \int_a^x e^{i\lambda\phi(t)}\,dt.$$

As we know, $F'(x) = e^{i\lambda\phi(x)}$ and hence $I(\lambda) = \int_a^b F'(x)\psi(x)\,dx$. If we integrate by parts, we see that

$$I(\lambda) = F(x)\psi(x)\big|_a^b - \int_a^b F(x)\psi'(x)\,dx = F(b)\psi(b) - \int_a^b F(x)\psi'(x)\,dx,$$

because $F(a) = 0$. Then, by Proposition A.6 we obtain

$$|F(x)| \leq c_k\lambda^{-1/k}, \quad \forall x \in (a,b),$$

so the triangle inequality gives

$$|I(\lambda)| \leq c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

□

A.3 In Higher Dimensions

Only some of the properties examined in Sections A.1 and A.2 can be generalised to higher dimensions. Fortunately, the localisation principle generalises. We will only state the result, since it follows from the one dimensional case. For more details the reader can check [11, Ch. VIII, §2].

Proposition A.10. *Suppose ϕ and ψ are smooth functions so that ψ has compact support. Assume also that ϕ has no critical points in the support of ψ. Then,*

$$I(\lambda) = O(\lambda^{-N}), \qquad \text{as } \lambda \to \infty$$

for every $N \in \mathbb{N}$.

B

Other Auxiliary Results

B.1 The Hardy-Littlewood-Sobolev Inequality

Proposition B.1. *Consider $f \in L^p(\mathbb{R}^n)$ and indices $0 < \gamma < n$, $1 < p < q < \infty$ such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}. \tag{B.1}$$

Then,

$$\|f * |y|^{-\gamma}\|_{L^q(\mathbb{R}^n)} \le A_{p,q}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The proof consists on splitting the integral

$$(f * |y|^{-\gamma})(x) = \int_{|y|<R} f(x-y)|y|^{-\gamma}\,dy + \int_{|y|>R} f(x-y)|y|^{-\gamma}\,dy$$

for some convenient $R > 0$ which will later be specified.

The first part will be treated by means of the Hardy-Littlewood maximal function. Observe that

$$\int_{|y|<R} f(x-y)|y|^{-\gamma}\,dy = f * |y|^{-\gamma}\chi_{B(0,R)}.$$

We notice that the function $|y|^{-\gamma}\chi_{B(0,R)}$ is nonnegative, radial and radially decreasing. Also,

$$\int_{|y|<R} |y|^{-\gamma}\,dy = \int_{\mathbb{S}^{n-1}}\int_0^R r^{n-1-\gamma}\,dr = C\frac{R^{n-\gamma}}{n-\gamma},$$

so it is integrable as long as $\gamma < n$. Hence, Lemma B.2 asserts that

$$f * |y|^{-\gamma}\chi_{B(0,R)}(x) \le C\frac{R^{n-\gamma}}{n-\gamma}Mf(x). \tag{B.2}$$

For the second integral, we use Hölder's inequality to write

$$|f*|y|^{-\gamma}\chi_{B(0,R)^c}| \le \|f\|_p \||y|^{-\gamma}\chi_{B(0,R)^c}(x)\|_{p'}, \tag{B.3}$$

where p and p' are conjugates. Observe that

$$\||y|^{-\gamma}\chi_{B(0,R)^c}(x)\|_{p'}^{p'} = \int_{|y|>R} |y|^{-\gamma p'}\,dy = C\int_R^\infty r^{n-1-\gamma p'}\,dr = \frac{C}{n-\gamma p'}R^{n-\gamma p'},$$

whenever $n-\gamma p' < 0$. A substitution of the value of γ using (B.1) gives the equality

$$\gamma p' - n = \frac{np'}{q},$$

which is positive because $q<\infty$. Hence, joining (B.2) and (B.3) we see that

$$|f*|y|^{-\gamma}(x)| \le C_{\gamma,p}\left(Mf(x)R^{n-\gamma} + \|f\|_p R^{n/p'-\gamma}\right).$$

Now choose $R>0$ so that

$$Mf(x)R^{n-\gamma} = \|f\|_p R^{n/p'-\gamma} \Leftrightarrow \frac{Mf(x)}{\|f\|_p} = R^{n/p'-n} = R^{-n/p}.$$

Hence, from condition (B.1),

$$\begin{aligned} |f*|y|^{-\gamma}(x)| &\le 2C(Mf(x))R^{n-\gamma} = 2C(Mf(x))\left(\frac{Mf(x)}{\|f\|_p}\right)^{-p\frac{n-\gamma}{n}} \\ &= 2C(Mf(x))\left(\frac{Mf(x)}{\|f\|_p}\right)^{-p\left(\frac{1}{p}-\frac{1}{q}\right)} = 2C(Mf(x))^{p/q}\|f\|_p^{1-p/q}. \end{aligned} \tag{B.4}$$

If we now estimate the L^q norm, we see that after renaming constants,

$$\int_{\mathbb{R}^n} |f*|y|^{-\gamma}(x)|^q\,dx \le C\|f\|_p^{q-p}\int_{\mathbb{R}^n} Mf(x)^p\,dx = C\|f\|_p^{q-p}\|Mf\|_p^p,$$

and since for $p>1$ the maximal function is a bounded $L^p\to L^p$ operator,

$$\int_{\mathbb{R}^n} |f*|y|^{-\gamma}(x)|^q\,dx \le C\|f\|_p^{q-p}\|f\|_p^p = C\|f\|_p^q,$$

so finally we get $\||f*|y|^{-\gamma}\|_q \le C\|f\|_p$, where the constant C depends on n,p,q and γ (and since these four are related by (B.1) we may remove the dependence on γ). □

The following lemma is which allows us to estimate the integral in $|y|<R$ in the proof above.

Lemma B.2. *Let ϕ be a nonnegative, integrable, radial and radially decreasing function in $\mathbb{R}^n$. Then, for every $f\in L^p(\mathbb{R}^n)$ with $1\le p<\infty$,*

$$|f*\phi(x)| \le Mf(x)\|\phi\|_1,$$

where $Mf(x)$ is the Hardy-Littlewood maximal function of f.

Proof. It is clear that we can assume $\|\phi\|_1 = 1$. The proof will be done by approximation. If we consider $s(x)$ to be a simple, positive, radial and radially decreasing function, so that

$$s(x) = \sum_{k=1}^{N} c_k \chi_{B(0,r_k)}(x), \quad r_k > 0,$$

then

$$\begin{aligned} s * f(x) &= \int_{\mathbb{R}^n} s(y) f(x-y)\,dy = \sum_{k=1}^{N} c_k \int_{B(0,r_k)} f(x-y)\,dy \\ &\leq \sum_{k=1}^{N} c_k |B(0,r_k)| Mf(x) = Mf(x)\|s\|_{L^1} \end{aligned} \tag{B.5}$$

and the result is satisfied. Now, the key fact is that every positive, integrable, radial, and radially decreasing function can be approximated by means of these simple functions. These functions are

$$s_n(x) = \sum_{k=1}^{4^{n-1}} \frac{1}{2^{n-1}} \chi_{\{x|\phi(x) > k/2^{n-1}\}}(x).$$

It is obvious that $s_{n+1}(x) \geq s_n(x)$, for all points x and every $n \in \mathbb{N}$. Observe that if $k/2^{n-1} \leq \phi(x) < (k+1)/2^{n-1}$, then s_n is summing precisely k terms in x so $s_n(x) = k/2^{n-1}$. Since while $n \to \infty$ the intervals are smaller, the more precise the estimation is, showing that $s_n \to \phi$ pointwise. Also, $s_n(x) \leq \phi(x)$ for every $n \in \mathbb{N}$, so $\|s_n\|_1 \leq \|\phi\|_1$. Once we see this, the monotone convergence theorem asserts that

$$\lim_{n\to\infty} s_n * f(x) = \phi * f(x),$$

and hence

$$\phi * f(x) = \lim_{n\to\infty} (s_n * f(x)) \leq Mf(x) \lim_{n\to\infty} \|s_n\|_{L^1} \leq Mf(x)\|\phi\|_{L^1}.$$

□

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